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EXTENDED JACOBSON DENSITY THEOREM FOR RINGS WITH DERIVATIONS AND AUTOMORPHISMS

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ABSTRACT

We introduce and study $\mathcal{M}$-outer derivations and automorphisms and prove a version of the Chevalley–Jacobson density theorem for rings with such derivations and automorphisms.

1. Introduction

In this paper we continue the project initiated recently in [10]; its main idea is to connect the concept of a dense action on modules with the concept of outerness of derivations and automorphisms.

In particular, one can view our results as generalizations of the Chevalley–Jacobson density theorem. This celebrated theorem is one of the important tools of ring theory and has already been generalized in various directions [1, 2, 3, 17, 24, 26, 29, 44, 54, 56, 57, 58] (see also [53, 15.7, 15.8] and [13, Extended Jacobson Density Theorem]).

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Perhaps the most natural and fundamental problem of the theory of derivations and automorphisms of rings and algebras is the question of their form, that is, the question of their innerness or outerness. This problem has a long and rich history. For instance, the classical Noether–Skolem theorem yields the solution of this problem for finite dimensional central simple algebras (see, e.g., [19, Section 4.3]). In another classical case of primitive rings with nonzero socle [23, Chapter IV] one already notes that it is sometimes convenient to extend the concept of innerness by allowing that the element, inducing the derivation or automorphism, lies in a bigger algebra. This is also the case in the theory of X-inner derivations and automorphisms initiated by Kharchenko (see, e.g., [4, 28, 39]). Finally, we remark that the theory of derivations and automorphisms does not play an important role only in ring theory, but also in functional analysis; concerning the question of innerness, an extensive and deep theory has been developed especially for derivations of C^*-algebras and commutative Banach algebras (see, e.g., [6, 40, 46] and also a more recent condensed survey [38]). Especially in analysis it is customary to treat derivations of one algebra into a bigger one (or into a bimodule); for most part of the paper we will work in this more general setting.

It is not our intention to address explicitly the question whether all derivations or automorphisms are inner on some special type of rings or algebras (though we hope that our results might turn out to be useful when considering this question). What we want to show here is that those derivations and automorphisms, which are "outer" in a sense defined below, have some very special and nice properties. To explain this more precisely, we have to introduce some notation and terminology.

In our main results we shall consider arbitrary rings. However, for the sake of simplicity we assume temporarily that $\mathcal{A}$ is an algebra over a field K and that $\mathcal{M}$ is a simple left $\mathcal{A}$-module such that $\mathrm{End}({}_{\mathcal{A}}\mathcal{M}) = K$. It is noteworthy that the latter condition is always satisfied for any complex Banach algebra $\mathcal{A}$ (namely, $\mathrm{End}({}_{\mathcal{A}}\mathcal{M})$ is the complex number field for any simple left $\mathcal{A}$-module $\mathcal{M}$ [6, Corollary 5, p. 128]). As both rings and algebras will be considered, it is perhaps necessary to mention that whenever we shall deal with a derivation or an automorphism of an algebra, we shall mean that it is a linear (and not just additive) map. Given $a \in \mathcal{A}$, it clearly defines a linear transformation L_a: $\mathcal{M}_K \to \mathcal{M}_K$ given by $L_a x = ax$ for all $x \in \mathcal{M}$. We shall say that a derivation d: $\mathcal{A} \to \mathcal{A}$ is $\mathcal{M}$-inner provided that there exists $T \in \mathrm{End}(\mathcal{M}_K)$ such that $L_{a^d} = [T, L_a]$ for all $a \in \mathcal{A}$ (derivations and automorphisms will be written as exponents); otherwise d is called $\mathcal{M}$-outer. Denote by $D(A)$ the set of all derivations of the algebra $\mathcal{A}$ and by $D_{\mathcal{M}}(\mathcal{A})$ the set of all $\mathcal{M}$-inner derivations of $\mathcal{A}$. Clearly $D(\mathcal{A})$ is a vector space over K and $D_{\mathcal{M}}(\mathcal{A})$ is a subspace of $D(\mathcal{A})$. Further, we shall say that automorphisms α and β of $\mathcal{A}$ are $\mathcal{M}$-dependent if there exists an invertible linear transformation T of $\mathcal{M}$ such that $L_{a^{\alpha^{-1}\beta}} = TL_aT^{-1}$ for all $a \in \mathcal{A}$; otherwise they are called $\mathcal{M}$-independent.

Recently the second author jointly with Šemrl proved the following result.

THEOREM 1.1 ([10, Theorem 3.6]): *Let $\mathcal{A}$ be a dense algebra of linear operators on a vector space $\mathcal{M}$, and let d be a derivation of $\mathcal{A}$. Then the following conditions are equivalent:*

(1) *d is $\mathcal{M}$-outer;*

(2) *given any linearly independent set $\{x_1, x_2, \dots, x_n\} \subseteq \mathcal{M}$ and arbitrary sets $\{y_1, y_2, \dots, y_n\}, \{z_1, z_2, \dots, z_n\} \subseteq \mathcal{M}$, there exists $a \in \mathcal{A}$ such that*

$$ax_i = y_i \quad \text{and} \quad a^d x_i = z_i \quad \text{for all } i = 1, 2, \dots, n.$$

A special case when $n = 2$ was obtained somewhat earlier [8]. We also mention results of Sinclair [47, Theorem 3.3] and Turovski and Shulman [50, Proposition 1.1] which do not really discover the connection between density and derivations, but they both establish that $\mathcal{M}$-outer derivations have some special properties; in a way, they can be considered as predecessors of Theorem 1.1. In the present paper we generalize this theorem to any ring with simple left module and compositions of derivations and automorphisms. In particular we prove the following theorem which is an important particular case of Theorem 5.3, the main result of the article.

THEOREM 1.2: *Let K be a field and $\mathcal{A}$ be a K-algebra with simple left module $\mathcal{M}$ such that* $\mathrm{End}(_{\mathcal{A}}\mathcal{M}) = K$. *Let k, l, m, n be positive integers, let $d_1, d_2, \dots, d_n \in D(\mathcal{A})$ and let $\alpha_1, \alpha_2, \dots, \alpha_l$ be automorphisms of $\mathcal{A}$. Let $\Delta_1, \Delta_2, \dots, \Delta_m$ be distinct words of the form $d_1^{i_1} d_2^{i_2} \cdots d_n^{i_n}$. Suppose that the following conditions are fulfilled:*

(1) *$d_1, d_2, \dots, d_n$ are linearly independent over K modulo $D_{\mathcal{M}}(\mathcal{A})$ (i.e., if $\lambda_i \in K$, then $\sum_{i=1}^n d_i\lambda_i$ is $\mathcal{M}$-inner if and only if each $\lambda_i = 0$);*

(2) *either $\mathrm{char}(K) = 0$ or $\mathrm{char}(K) = p > 0$ and each $i_t < p$;*

(3) *α_i and α_j are $\mathcal{M}$-independent for all $i \neq j$.*

Then for any linearly independent elements $x_1, x_2, \dots, x_k \in \mathcal{M}$ and for any elements $z_{ijt} \in \mathcal{M}$, $i = 1, 2, \dots, k$, $j = 1, 2, \dots, m$, $t = 1, 2, \dots, l$, there exists $a \in \mathcal{A}$ with

$$a^{\Delta_j \alpha_t} x_i = z_{ijt} \quad \text{for all } i = 1, 2, \dots, k,\ j = 1, 2, \dots, m,\ t = 1, 2, \dots, l.$$

(It is understood that $d_1^0 d_2^0 \cdots d_n^0$ acts as the identity map on $\mathcal{A}$.)

Our approach to the proof of this (and other) theorems was inspired by classical results on derivations and automorphisms of primitive rings with nonzero socle (see [23, Chapter IV] and [4, Chapter 4]) and is different from methods used in [10]. It is based on the detailed study of the concepts of $\mathcal{M}$-outer derivations and automorphisms and their interpretations in terms of associated modules.

The paper is organized as follows. Section 2 is preliminary. In Sections 3 and 4 we study automorphisms and derivations, respectively. As we shall see, the proofs of results on automorphisms are considerably easier and are basically simple consequences of the Extended Jacobson Density Theorem (see Proposition 2.1). The main result is proved in Section 5. A somewhat sharper version of the main result for primitive rings is obtained in Section 6. The goal of Section 7 is to prove some applications of our results concerning derivations, and thereby to illustrate the new techniques that can now be used in the study of derivations.

Our results are reminiscent of Kharchenko's freeness theorem [27], but actually they refer to the distinct classes of rings. Kharchenko's result is connected with primitive rings with nonzero socle, while our results are vacuous in this setting because these rings do not admit $\mathcal{M}$-outer derivations and automorphisms (see Corollaries 3.3 and 4.3). Nevertheless, Kharchenko's theory and our results have something in common: they both make it possible for one to reduce certain problems on general derivations and automorphisms to "inner" (that is, X-inner or $\mathcal{M}$-inner) ones. Both theories certainly have their limitations: one must have some identity satisfied by derivations or automorphisms in a (semi)prime ring in order to apply Kharchenko's theory, while our results give some information only modulo the Jacobson radical. In a way they are complementary to each other and, roughly speaking, they both give the same, somewhat surprising message: it is easier to deal with "outer" maps than with "inner" ones.

2. Density and local modules

Let $\mathcal{A}$ be a ring. Recall that a left $\mathcal{A}$-module $\mathcal{M}$ is called **simple** if $\mathcal{A}\mathcal{M} \neq 0$ and ${}_{\mathcal{A}}\mathcal{M}$ has no nonzero proper submodules. If $\mathcal{M}$ is simple, then by Schur's Lemma, $\mathrm{End}({}_{\mathcal{A}}\mathcal{M})$ is a division ring. The following result follows easily from the density theorem for semisimple modules and, on the other hand, is just a rewording of [13, Extended Jacobson Density Theorem].

PROPOSITION 2.1: *Let $\mathcal{A}$ be a ring, let $n, m_1, m_2, \ldots, m_n$ be positive integers, let $\mathcal{M}_i$, $i = 1, 2, \ldots, n$, be pairwise nonisomorphic simple left $\mathcal{A}$-modules and let $\mathcal{D}_i = \mathrm{End}({}_{\mathcal{A}}\mathcal{M}_i)$. Further, let $x_{i1}, x_{i2}, \ldots, x_{im_i} \in \mathcal{M}_i$ be linearly independent* *over $\mathcal{D}_i$ and let $y_{i1}, y_{i2}, \dots, y_{im_i} \in \mathcal{M}_i$, $i = 1, 2, \dots, n$. Then there exists $a \in \mathcal{A}$ such that $ax_{ij} = y_{ij}$ for all $i = 1, 2, \dots, n$, $j = 1, 2, \dots, m_i$.*

Recall that a left $\mathcal{A}$-module $\mathcal{M}$ is said to be **local**, if it contains a unique maximal submodule $\mathcal{L}$ such that $\mathcal{M}/\mathcal{L}$ is a simple module and every proper submodule of $\mathcal{M}$ is contained in $\mathcal{L}$. Note that every simple left $\mathcal{A}$-module is local.

We also recall that the Jacobson radical $J(\mathcal{K})$ of a left $\mathcal{A}$-module $\mathcal{K}$ is equal to the intersection of all submodules $\mathcal{U}$ of $\mathcal{K}$ such that $\mathcal{K}/\mathcal{U}$ is a simple $\mathcal{A}$-module. If $\mathcal{K}$ has no such submodules, then $J(\mathcal{K}) = \mathcal{K}$.

Remark 2.2: Let $\mathcal{M}$ be an $\mathcal{A}$-module with submodule $\mathcal{L}$ and let $\mathcal{N}$, $\mathcal{N}_i$, $1 \le i \le n$, be left $\mathcal{A}$-modules. Then:

(1) *if $\alpha\colon \mathcal{M} \to \mathcal{N}$ is an epimorphism of modules, then $J(\mathcal{M})\alpha \subseteq J(\mathcal{N})$;*
(2) *$J(\bigoplus_{i=1}^n \mathcal{N}_i) = \bigoplus_{i=1}^n J(\mathcal{N}_i)$;*
(3) *assume that $\mathcal{M}$ has a generating set $\{x_1, x_2, \dots, x_n\}$ such that each $x_i \in \mathcal{A}x_i$, and let $\mathcal{L}$ be a submodule of $\mathcal{M}$ such that $\mathcal{L} + J(\mathcal{M}) = \mathcal{M}$; then $\mathcal{L} = \mathcal{M}$ (Nakayama's Lemma);*
(4) *$\mathcal{M}$ is local with maximal submodule $\mathcal{L}$ if and only if $\mathcal{A}x = \mathcal{M}$ for all $x \in \mathcal{M} \setminus \mathcal{L}$; if $\mathcal{M}$ is a local module with maximal submodule $\mathcal{L}$, then $J(\mathcal{M}) = \mathcal{L}$.*

Since these statements are well-known for unital modules over a ring with unity and we do not assume that $\mathcal{A}$ has a unity, we shall sketch their proofs.

Proof: (1) Clearly, if $\alpha\colon \mathcal{M} \to \mathcal{N}$ is an epimorphism of modules and $\mathcal{K}$ is a submodule of $\mathcal{N}$ such that $\mathcal{N}/\mathcal{K}$ is a simple left $\mathcal{A}$-module, then $\mathcal{M}/\mathcal{K}\alpha^{-1}$ is a simple left $\mathcal{A}$-module (isomorphic to $\mathcal{N}/\mathcal{K}$). Therefore $J(\mathcal{M}) \subseteq J(\mathcal{N})\alpha^{-1}$ and so $J(\mathcal{M})\alpha \subseteq J(\mathcal{N})$.

(2) Applying (1) to canonical projections $\bigoplus_{i=1}^n \mathcal{N}_i \to N_j$, one easily gets that $J(\bigoplus_{i=1}^n \mathcal{N}_i) \subseteq \bigoplus_{i=1}^n J(\mathcal{N}_i)$. On the other hand, if $\gamma\colon \bigoplus_{i=1}^n \mathcal{N}_i \to \mathcal{K}$ is an epimorphism onto a simple module $\mathcal{K}$, then clearly $J(\mathcal{N}_i) \subseteq \ker(\gamma)$ for all i, which completes the proof.

(3) Assume that $\mathcal{L} \ne \mathcal{M}$. Since $\mathcal{M}$ is finitely generated, by Zorn's Lemma there exists a maximal submodule $\mathcal{N}$ of $\mathcal{M}$ containing $\mathcal{L}$. Clearly $x_i \notin \mathcal{N}$ for some i and so $\mathcal{A}x_i \not\subseteq \mathcal{N}$. It follows that $\mathcal{M}/\mathcal{N}$ is a simple module and hence $J(\mathcal{M}) \subseteq \mathcal{N}$, contradicting $\mathcal{L} + J(\mathcal{M}) = \mathcal{M}$.

(4) Suppose that $\mathcal{M}$ is local with maximal submodule $\mathcal{L}$ and $x \in \mathcal{M} \setminus \mathcal{L}$. Since $\mathcal{M}/\mathcal{L}$ is a simple $\mathcal{A}$-module, $\mathcal{A}x \not\subseteq \mathcal{L}$ and so $\mathcal{A}x = \mathcal{M}$. On the other hand, if $\mathcal{A}x = \mathcal{M}$ for all $x \in \mathcal{M} \setminus \mathcal{L}$, then $\mathcal{M}/\mathcal{L}$ is a simple left $\mathcal{A}$-module and every proper submodule of $\mathcal{M}$ is contained in $\mathcal{L}$. In particular $J(\mathcal{M}) = \mathcal{L}$. ∎

THEOREM 2.3: *Let $\mathcal{A}$ be a ring, let $n, m_1, m_2, \ldots, m_n$ be positive integers, let $\mathcal{M}_i$ be a local left $\mathcal{A}$-module with a maximal submodule $\mathcal{L}_i$, $i = 1, 2, \ldots, n$, let $\mathcal{N}_i = \mathcal{M}_i/\mathcal{L}_i$ and let $\mathcal{D}_i = \mathrm{End}({}_{\mathcal{A}}\mathcal{N}_i)$. Further, let $x_{i1}, x_{i2}, \ldots, x_{im_i}$ be elements of $\mathcal{M}_i$ linearly independent over $\mathcal{D}_i$ modulo $\mathcal{L}_i$, $i = 1, 2, \ldots, n$. Suppose that $\mathcal{N}_i \not\cong \mathcal{N}_j$ for all $1 \le i \ne j \le n$. Then for any $y_{i1}, y_{i2}, \ldots, y_{im_i} \in \mathcal{M}_i$ there exists $a \in \mathcal{A}$ such that $ax_{ij} = y_{ij}$ for all $1 \le i \le n$, $1 \le j \le m_i$.*

Proof: Set $\mathcal{M} = \bigoplus_{i=1}^n (\mathcal{M}_i^{m_i})$ where $\mathcal{M}_i^{m_i}$ is the direct sum of m_i copies of $\mathcal{M}_i$. Let

$$\overline{x} = (x_{11}, \ldots, x_{1m_1}, x_{21}, \ldots, x_{2m_2}, \ldots, x_{n1}, \ldots, x_{nm_n}) \quad \text{and}$$
$$\overline{y} = (y_{11}, \ldots, y_{1m_1}, y_{21}, \ldots, y_{2m_2}, \ldots, y_{n1}, \ldots, y_{nm_n}).$$

We claim that $\mathcal{A}\overline{x} = \mathcal{M}$. Indeed, by Remark 2.2(3) it is enough to show that $\mathcal{A}\overline{x} + J(\mathcal{M}) = \mathcal{M}$. According to Remark 2.2(2), $J(\mathcal{M}) = \bigoplus_{i=1}^n (J(\mathcal{M}_i)^{m_i})$ and so $\mathcal{M}/J(\mathcal{M}) = \bigoplus_{i=1}^n [\mathcal{M}_i/J(\mathcal{M}_i)]^{m_i}$. It now follows from Proposition 2.1 that $\mathcal{A}(\overline{x} + J(\mathcal{M})) = \mathcal{M}/J(\mathcal{M})$ and so $\mathcal{A}\overline{x} + J(\mathcal{M}) = \mathcal{M}$. Therefore $\mathcal{A}\overline{x} = \mathcal{M}$. In particular there exists $a \in \mathcal{A}$ with $a\overline{x} = \overline{y}$. Thus $ax_{ij} = y_{ij}$ for all i and j. ∎

3. Density and $\mathcal{M}$-outer automorphisms

In what follows $\mathcal{A}$ will be a ring with simple left module $\mathcal{M}$. We set $\mathcal{D} = \mathrm{End}({}_{\mathcal{A}}\mathcal{M})$. By Schur's Lemma $\mathcal{D}$ is a division ring. Given $a \in \mathcal{A}$, we define a map $L_a \colon \mathcal{M} \to \mathcal{M}$ by $L_a x = ax$ for all $x \in \mathcal{M}$. Clearly $L_a \in \mathrm{End}(\mathcal{M}_{\mathcal{D}})$.

Consider the ring $\mathrm{End}(\mathcal{M})$ of all endomorphisms of the additive group $\mathcal{M}$ acting on $\mathcal{M}$ from the left and consider the ring $\mathrm{End}(\mathcal{M}_{\mathcal{D}})$ as a subring of $\mathrm{End}(\mathcal{M})$. Given $u \in \mathcal{D}$, we define a map $R_u \colon \mathcal{M} \to \mathcal{M}$ by $R_u x = xu$, $x \in \mathcal{M}$. Clearly $\mathcal{D}^o = \{R_u \colon u \in \mathcal{D}\}$ is a subdivision ring of $\mathrm{End}(\mathcal{M})$ anti-isomorphic to $\mathcal{D}$. Note that $[R_u, L_a] = 0$ for all $a \in \mathcal{A}$ and $u \in \mathcal{D}$. Clearly

$$\mathcal{D}^o = \{\Lambda \in \mathrm{End}(\mathcal{M}) \colon [\Lambda, L_a] = 0 \quad \text{for all } a \in \mathcal{A}\} \tag{1}$$

(see [4, p. 129]).

Let τ be an automorphism of $\mathcal{D}$. Recall that an invertible element $T \in \mathrm{End}(\mathcal{M})$ is called a τ-semilinear automorphism of the right vector space $\mathcal{M}_{\mathcal{D}}$ provided that $T(xu) = (Tx)u^\tau$ for all $x \in \mathcal{M}$ and $u \in \mathcal{D}$. Clearly an invertible element $T \in \mathrm{End}(\mathcal{M})$ is τ-semilinear if and only if $TR_uT^{-1} = R_{u^\tau}$ for all $u \in \mathcal{D}$. Further, an invertible element $T \in \mathrm{End}(\mathcal{M})$ is called a semilinear automorphism of the right vector space $\mathcal{M}_{\mathcal{D}}$ if it is τ-semilinear for some automorphism τ of $\mathcal{D}$. Clearly, an invertible element $T \in \mathrm{End}(\mathcal{M})$ is a semilinear automorphism of $\mathcal{M}_{\mathcal{D}}$ if and only if $T\mathcal{D}^oT^{-1} = \mathcal{D}^o$.

Now we introduce the concepts of $\mathcal{M}$-inner and $\mathcal{M}$-outer automorphisms.

Definition 3.1: An automorphism α of the ring $\mathcal{A}$ is called **$\mathcal{M}$-inner** if there exist an invertible element $T \in \mathrm{End}(\mathcal{M})$ such that

$$TL_aT^{-1} = L_{a^\alpha} \quad \text{for all } a \in \mathcal{A}; \tag{2}$$

otherwise it is called **$\mathcal{M}$-outer**.

Suppose that (2) is satisfied. We claim that then T is a semilinear automorphism of $\mathcal{M}_\mathcal{D}$. Indeed, let $u \in \mathcal{D}$. Then $[R_u, L_a] = 0$ for all $a \in \mathcal{A}$ and so

$$0 = T[R_u, L_a]T^{-1} = [TR_uT^{-1}, TL_aT^{-1}] = [TR_uT^{-1}, L_{a^\alpha}].$$

Therefore $[TR_uT^{-1}, L_{a^\alpha}] = 0$ for all $a \in \mathcal{A}$. Since α is an automorphism of $\mathcal{A}$, we conclude that $[TR_uT^{-1}, L_a] = 0$ for all $a \in \mathcal{A}$ and whence $TR_uT^{-1} \in \mathcal{D}^o$ by (1). Therefore $T\mathcal{D}^oT^{-1} \subseteq \mathcal{D}^o$. Substituting $a^{\alpha^{-1}}$ for a in (2), we get $TL_{a^{\alpha^{-1}}}T^{-1} = L_a$ and so $T^{-1}L_aT = L_{a^{\alpha^{-1}}}$ for all $a \in \mathcal{A}$. By the above result $T^{-1}\mathcal{D}^oT \subseteq \mathcal{D}^o$ and whence $T^{-1}\mathcal{D}^oT = \mathcal{D}^o$. Thus T is a semilinear automorphism of the right vector space $\mathcal{M}_\mathcal{D}$.

The following result is a particular case of [23, Isomorphism Theorem, p. 79].

THEOREM 3.2: *Let $\mathcal{A}$ be a primitive ring with nonzero socle and let $\mathcal{M}$ be a faithful simple left $\mathcal{A}$-module. Then every automorphism of $\mathcal{A}$ is $\mathcal{M}$-inner.*

Suppose that $\dim_\mathcal{D}(\mathcal{M}) = n < \infty$ where $\mathcal{M}$ is a faithful simple left $\mathcal{A}$-module and $\mathcal{D} = \mathrm{End}({}_\mathcal{A}\mathcal{M})$. The Jacobson Density Theorem then implies that $\mathcal{A}$ is isomorphic to the ring of $n \times n$ matrices over $\mathcal{D}$, so that the socle of $\mathcal{A}$ is nonzero. Therefore, as an immediate corollary to Theorem 3.2 we can state

COROLLARY 3.3: *Let $\mathcal{A}$ be a primitive ring with faithful simple left module $\mathcal{M}$ and let $\mathcal{D} = \mathrm{End}({}_\mathcal{A}\mathcal{M})$. Suppose that $\mathcal{A}$ has an $\mathcal{M}$-outer automorphism. Then the socle of $\mathcal{A}$ is equal to 0 and $\dim_\mathcal{D}(\mathcal{M}) = \infty$.*

It is not too difficult to find primitive rings with trivial socle having $\mathcal{M}$-outer automorphisms. A more nontrivial question is whether the $\mathcal{M}$-outerness depends only on the automorphism or also on the module M. The answer is

Example 3.4: Let F be a field of characteristic zero and $\mathcal{A}$ be the Weyl algebra over F generated by x and y and the relation $[x, y] = 1$. Recall that every element $p(x, y)$ of $\mathcal{A}$ can be uniquely written in the form

$$p(x,y) = \sum_{i=0}^{m}\sum_{j=0}^{n} a_{ij}x^iy^j = \sum_{j=0}^{n}\sum_{i=0}^{m} b_{ji}y^jx^i$$

where each $a_{ij}, b_{ji} \in F$ and $a_{mn} \neq 0 \neq b_{nm}$ (see [37, p. 19]). Further, $F[x]$, the algebra of polynomials in x over F, is a faithful simple left $\mathcal{A}$-module under the multiplication given by

$$p \cdot f = \left(\sum_{i=0}^{m} \sum_{j=0}^{n} a_{ij} x^i y^j \right) \cdot f(x) = \sum_{i=0}^{m} \sum_{j=0}^{n} a_{ij} x^i \frac{d^j f(x)}{dx^j}, \quad p \in \mathcal{A}, f \in F[x]$$

(i.e., $x \cdot f(x) = xf(x)$ and $y \cdot f(x) = f'(x)$). Analogously, $F[y]$ is a faithful simple left $\mathcal{A}$-module with respect to

$$p \cdot h = \left(\sum_{j=0}^{n} \sum_{i=0}^{m} b_{ji} y^j x^i \right) \cdot h(y) = \sum_{j=0}^{n} \sum_{i=0}^{m} (-1)^i b_{ji} y^j \frac{d^i h(y)}{dy^i}, \quad p \in \mathcal{A}, h \in F[y]$$

(i.e., $y \cdot h(y) = yh(y)$ and $x \cdot h(y) = -h'(y)$).

We now define an automorphism α of the algebra $\mathcal{A}$ by the rule $x^\alpha = x + 1$, $y^\alpha = y$. Since $[x^\alpha, y^\alpha] = 1$, α is a well-defined automorphism of $\mathcal{A}$. Set $\mathcal{M} = F[x]$ and define a linear operator T: $\mathcal{M} \to \mathcal{M}$ by the rule $Tf(x) = f(x+1)$, $f \in \mathcal{M}$. Clearly T is a bijective linear operator with $T^{-1}f(x) = f(x-1)$. One can easily check that $TL_xT^{-1} = L_{x^\alpha}$ and $TL_yT^{-1} = L_{y^\alpha}$, and so $TL_pT^{-1} = L_{p^\alpha}$ for all $p \in \mathcal{A}$. Thus α is $\mathcal{M}$-inner.

On the other hand, we claim that α is $\mathcal{N}$-outer, where $\mathcal{N} = F[y]$. Indeed, suppose that there is an invertible operator S: $F[y] \to F[y]$ such that $L_{x+1} = L_{x^\alpha} = SL_xS^{-1}$ and $L_y = L_{y^\alpha} = SL_yS^{-1}$. Then $S^{-1}L_x + S^{-1} = L_xS^{-1}$. Since $L_x 1 = -1' = 0$ (here $1 \in F[y]$), it follows that $S^{-1}1 = L_x \cdot S^{-1}1$, that is, $h(y) = -h(y)'$ where $h(y) = S^{-1}1$. Thus $S^{-1}1 = h(y) = 0$, contradicting the bijectivity of S. Therefore, α is $\mathcal{N}$-outer.

We have thereby showed that even on simple rings there can exist automorphisms which are simultaneously $\mathcal{M}$-inner and $\mathcal{N}$-outer for some simple left modules $\mathcal{M}$ and $\mathcal{N}$.

Suppose for a while that $\mathcal{A}$ is a prime ring. Recall that an automorphism α of the ring $\mathcal{A}$ is said to be X-inner if there exists an invertible element $t \in Q_s(\mathcal{A})$, the symmetric ring of quotients of $\mathcal{A}$, such that $x^\alpha = txt^{-1}$ for all $x \in \mathcal{A}$ (see [4, Chapter 2]). If $\mathcal{A}$ is a primitive ring with faithful simple left $\mathcal{A}$-module $\mathcal{M}$, then $\mathcal{M}$ is also a faithful simple left $Q_s(\mathcal{A})$-module (see [52], [4, Theorem 4.1.1]). Therefore every X-inner automorphism of $\mathcal{A}$ is also $\mathcal{M}$-inner. Example 3.4 shows that the converse is not true. Namely, the automorphism α in this example is $\mathcal{M}$-inner but X-outer since it is $\mathcal{N}$-outer. Moreover, an automorphism can be $\mathcal{M}$-inner for any simple faithful simple left module $\mathcal{M}$, but still X-outer. In view of Theorem 3.2 this is true for any X-outer automorphism of a primitive ring with nonzero socle.

Now, again let $\mathcal{A}$ be any ring with a simple left module $\mathcal{M}$ and α be an automorphism of $\mathcal{A}$. We define a left $\mathcal{A}$-module $\mathcal{M}_\alpha$ as follows. As an abelian group, $\mathcal{M}_\alpha = \mathcal{M}$. Given $a \in \mathcal{A}$ and $x \in \mathcal{M}_\alpha$, we set $a *_\alpha x = a^\alpha x$. Clearly $\mathcal{M}_\alpha$ is a simple left $\mathcal{A}$-module. Further,

$$(a *_\alpha x)u = (a^\alpha x)u = a^\alpha(xu) = a *_\alpha (xu)$$

for all $u \in \mathcal{D}$, $a \in \mathcal{A}$ and $x \in \mathcal{M}_\alpha$. Therefore there exists a monomorphism of rings $\mathcal{D} \to \mathrm{End}({}_\mathcal{A}\mathcal{M}_\alpha)$. Since ${}_\mathcal{A}M \cong {}_\mathcal{A}(M_\alpha)_{\alpha^{-1}}$, we conclude that $\mathcal{D} \cong \mathrm{End}({}_\mathcal{A}\mathcal{M}_\alpha)$. Henceforward we shall identify them, i.e.,

$$\mathrm{End}({}_\mathcal{A}\mathcal{M}_\alpha) = \mathcal{D}. \tag{3}$$

We shall say that automorphisms α and β of $\mathcal{A}$ are **$\mathcal{M}$-independent** if the automorphism $\alpha^{-1}\beta$ (and hence $\beta^{-1}\alpha$) is $\mathcal{M}$-outer; otherwise they are called $\mathcal{M}$-dependent.

PROPOSITION 3.5: *Let $\mathcal{A}$ be a ring with simple left module $\mathcal{M}$ and automorphisms α and β. Then α and β are $\mathcal{M}$-dependent if and only if the left $\mathcal{A}$-modules $\mathcal{M}_\alpha$ and $\mathcal{M}_\beta$ are isomorphic.*

Proof: Assume that α and β are $\mathcal{M}$-dependent. Then there exists an automorphism T of the additive group $\mathcal{M}$ such that $L_{a^{\alpha^{-1}\beta}} = TL_aT^{-1}$ for all $a \in \mathcal{A}$. Therefore $L_{a^\beta} = TL_{a^\alpha}T^{-1}$ and so

$$L_{a^\beta}T = TL_{a^\alpha} \quad \text{for all } a \in \mathcal{A}. \tag{4}$$

Consider T as a bijective additive map $\mathcal{M}_\alpha \to \mathcal{M}_\beta$. Then (4) implies that T is an isomorphism of left $\mathcal{A}$-modules.

Conversely, let $T\colon \mathcal{M}_\alpha \to \mathcal{M}_\beta$ be an isomorphism of left $\mathcal{A}$-modules. Then

$$T(L_{a^\alpha}x) = T(a *_\alpha x) = a *_\beta (Tx) = L_{a^\beta}(Tx) \quad \text{for all } a \in \mathcal{A},\ x \in \mathcal{M}_\beta.$$

That is, $TL_{a^\alpha} = L_{a^\beta}T$. Substituting $a^{\alpha^{-1}}$ for a, we get $TL_a = L_{a^{\alpha^{-1}\beta}}T$ and so $TL_aT^{-1} = L_{a^{\alpha^{-1}\beta}}$ for all $a \in \mathcal{A}$. Thus $\alpha^{-1}\beta$ is $\mathcal{M}$-inner. ∎

The main result of this section is

THEOREM 3.6: *Let $\mathcal{A}$ be a ring with simple module ${}_\mathcal{A}\mathcal{M}$, let $\mathcal{D} = \mathrm{End}({}_\mathcal{A}\mathcal{M})$ and let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be automorphisms of $\mathcal{A}$. The following conditions are equivalent:*

(1) *α_i and α_j are $\mathcal{M}$-independent for all $i \neq j$;*
(2) *given any $x_1, x_2, \ldots, x_m \in \mathcal{M}$ linearly independent over $\mathcal{D}$ and any $y_{ij} \in \mathcal{M}$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, there exists $a \in \mathcal{A}$ such that $a^{\alpha_i} x_j = y_{ij}$ for all $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$.*

Proof: From Propositions 3.5 and 2.1 it follows that (1) implies (2). To prove the converse, suppose that two of the automorphisms, $\alpha = \alpha_i$ and $\beta = \alpha_j$, $i \neq j$, are $\mathcal{M}$-dependent. Thus, (4) holds for some invertible $T \in \mathrm{End}(\mathcal{M})$. But then $a^{\beta} Tx = 0$, $x \in M$, implies $a^{\alpha} x = 0$, and so (2) certainly does not hold. ∎

Let us mention that Theorem 3.2 follows easily from Theorem 3.6. Indeed, let $\mathcal{A}$ be a primitive ring with nonzero socle, α be an automorphism of $\mathcal{A}$, and $\mathcal{M}$ be a faithful simple left $\mathcal{A}$-module. Pick any nonzero $a \in \mathcal{A}$ such that $a\mathcal{M} = x_0\mathcal{D}$ for some nonzero $x_0 \in \mathcal{M}$. Next pick any $0 \neq x_1 \in a^{\alpha}\mathcal{M}$. If $b \in \mathcal{A}$ is any element such that $bx_0 = 0$, then $ba = 0$ and hence $b^{\alpha}a^{\alpha} = 0$. Therefore $b^{\alpha}x_1 = 0$. But then Theorem 3.6 tells us that α cannot be $\mathcal{M}$-outer.

On the other hand, Theorem 3.2 can be deduced at once from Proposition 3.5 together with the fact that primitive rings with nonzero socle do not have nonisomorphic faithful simple left modules.

4. Density and $\mathcal{M}$-outer derivations

Throughout this section, $\mathcal{A}$ will be a ring with simple left module $\mathcal{M}$, $\mathcal{D} = \mathrm{End}({}_{\mathcal{A}}\mathcal{M})$ and F will be the prime subfield of the division ring $\mathcal{D}$. Clearly $\mathcal{M}$ is a vector space over F. Recall that an additive mapping $T\colon \mathcal{M} \to \mathcal{M}$ is called a differential transformation if there exists a map $\gamma\colon \mathcal{D} \to \mathcal{D}$ such that $T(xu) - (Tx)u = xu^{\gamma}$ for all $x \in \mathcal{M}$, $u \in \mathcal{D}$. It is easy to see that γ is a derivation of $\mathcal{D}$.

By a derivation $d\colon \mathcal{A} \to \mathrm{End}(\mathcal{M}_{\mathcal{D}})$ we mean an additive map satisfying $(xy)^d = L_x y^d + x^d L_y$ for all $x, y \in A$. This extends the usual concept of a derivation of a ring into itself since any derivation $d\colon \mathcal{A} \to \mathcal{A}$ gives rise to a derivation $\overline{d}\colon \mathcal{A} \to \mathrm{End}(\mathcal{M}_{\mathcal{D}})$ given by $a^{\overline{d}} = L_{a^d}$. Moreover, if the ring $\mathcal{A}$ is primitive and $\mathcal{M}$ is a faithful $\mathcal{A}$-module, we can consider $\mathcal{A}$ as a subring of $\mathrm{End}(\mathcal{M}_{\mathcal{D}})$ via the embedding $x \mapsto L_x$.

Definition 4.1: A derivation $d\colon \mathcal{A} \to \mathrm{End}(\mathcal{M}_{\mathcal{D}})$ is called **$\mathcal{M}$-inner** if there exists an element $T \in \mathrm{End}(\mathcal{M})$ such that

$$(5) \qquad [T, L_a] = a^d \quad \text{for all } a \in \mathcal{A};$$

otherwise it is called **$\mathcal{M}$-outer**.

Of course, if d is a derivation of $\mathcal{A}$ into itself, then we shall say that d is $\mathcal{M}$-inner if $\overline{d}$: $\mathcal{A} \to \mathrm{End}(\mathcal{M}_{\mathcal{D}})$ is $\mathcal{M}$-inner.

Note that (5) yields

$$0 = [T, [R_u, L_a]] = [R_u, [T, L_a]] + [[T, R_u], L_a] = [L_a, [R_u, T]]$$

for all $u \in \mathcal{D}$ and $a \in \mathcal{A}$ because $[R_u, L_a] = 0 = [R_u, a^d]$. By (1), $[R_u, T] \subseteq \mathcal{D}^o$. Setting $[R_u, T] = R_{u'}$, we see that $T(xu) - (Tx)u = xu'$ for all $x \in \mathcal{M}$. That is to say, T is a differential transformation of $\mathcal{M}$.

We begin with some results analogous to those obtained in the precedent section. First we have

THEOREM 4.2: *Let $\mathcal{A}$ be a primitive ring with nonzero socle and let $\mathcal{M}$ be a faithful simple left $\mathcal{A}$-module. Then every derivation d: $\mathcal{A} \to \mathrm{End}(\mathcal{M}_{\mathcal{D}})$ is $\mathcal{M}$-inner.*

Proof: Since $\mathcal{A}$ is a primitive ring with nonzero socle, its maximal right ring of quotients Q_{mr} is equal to $\mathrm{End}(\mathcal{M}_{\mathcal{D}})$ [4, Theorem 4.3.7]. Arguing as in the proof of [4, Proposition 2.5.1], one can easily show that every derivation d: $\mathcal{A} \to Q_{mr}$ uniquely extends to a derivation $\tilde{d}$: $Q_{mr} \to Q_{mr}$. Clearly d is $\mathcal{M}$-inner if and only if $\tilde{d}$ is $\mathcal{M}$-inner. The result now follows from [23, Theorem 3, p. 87]. ∎

As in the automorphism case this yields

COROLLARY 4.3: *Let $\mathcal{A}$ be a primitive ring with faithful simple left module $\mathcal{M}$ and let $\mathcal{D} = \mathrm{End}({}_{\mathcal{A}}\mathcal{M})$. Suppose that $\mathcal{A}$ has an $\mathcal{M}$-outer derivation. Then the socle of $\mathcal{A}$ is equal to* 0 *and* $\dim_{\mathcal{D}}(\mathcal{M}) = \infty$.

Next we consider an analogue of Example 3.4.

Example 4.4: Let F be a field of characteristic 0 and let $\mathcal{A}$ be the Weyl algebra generated by x_i, y_i, $i = 1, 2, \ldots$ and the relations

$$[y_i, x_j] = \delta_{ij}, \quad [x_i, x_j] = 0 \quad \text{and} \quad [y_i, y_j] = 0 \quad \text{for all } i, j.$$

Let $\mathcal{M} = F[x_1, x_2, \ldots]$ be the algebra of polynomials in $x_1, x_2, \ldots$. Given $f \in \mathcal{M}$, we define $x_i \cdot f = x_i f$ and $y_i \cdot f = \partial f / \partial x_i$. Extending this multiplication to $\mathcal{A}$, $\mathcal{M}$ becomes a faithful simple left $\mathcal{A}$-module.

Similarly, $\mathcal{N} = F[y_1, y_2, \ldots]$ becomes a faithful simple left $\mathcal{A}$-module by defining $y_i \cdot f = -y_i f$ and $x_i \cdot f = \partial f / \partial y_i$.

Define a derivation d: $\mathcal{A} \to \mathcal{A}$ by $x_i^d = 1$, $y_i^d = 0$ and extend it to $\mathcal{A}$ according to the derivation law. Note that d is well-defined for

$$[y_i^d, x_j] + [y_i, x_j^d] = [x_i^d, x_j] + [x_i, x_j^d] = [y_i^d, y_j] + [y_i, y_j^d] = 0,$$

$i, j = 1, 2, \ldots$.

Define a linear operator $T\colon \mathcal{M} \to \mathcal{M}$ by $Tf = \partial f/\partial x_1 + \partial f/\partial x_2 + \cdots$. Note that T is well-defined and that $L_{p^d} = [T, L_p]$ for all $p \in \mathcal{A}$. Thus, d is $\mathcal{M}$-inner.

Finally, suppose that there is an operator $S\colon \mathcal{N} \to \mathcal{N}$ such that $L_{p^d} = [S, L_p]$ for all $p \in \mathcal{A}$. In particular, we have $[S, L_{x_i}]\ 1 = 1$, $i = 1, 2, \ldots$, which means that $h = S1 \in \mathcal{N}$ satisfies $\partial h/\partial y_i = -1$, $i = 1, 2, \ldots$. This is clearly impossible and so d is $\mathcal{N}$-outer.

Let $\mathcal{A}$ be a primitive ring. Repeating the arguments from the preceding section we see that an X-inner derivation $d\colon \mathcal{A} \to \mathcal{A}$ (i.e., such that $x^d = tx - xt$, $x \in \mathcal{A}$ for some $t \in Q_s(\mathcal{A})$) is also $\mathcal{M}$-inner for any faithful simple left $\mathcal{A}$-module $\mathcal{M}$, and that the converse is not true in general.

Let $d\colon \mathcal{A} \to \mathrm{End}(\mathcal{M}_{\mathcal{D}})$ be a derivation, let V be a vector space over F with basis $\{d, e\}$. Set $\mathcal{M}_d = V \otimes_F \mathcal{M}$ and define a left $\mathcal{A}$-module structure on the vector space $\mathcal{M}_d$ as follows:

$$a(d \otimes x + e \otimes y) = d \otimes ax + e \otimes (a^d x + ay) \quad \text{for all } x, y \in M,\ a \in \mathcal{A}.$$

Clearly $\mathcal{L}(d) = e \otimes \mathcal{M}$ is a submodule of $\mathcal{M}_d$ and $\mathcal{M}_d/\mathcal{L}(d) \cong \mathcal{M}$ via the map $d \otimes x + \mathcal{L}(d) \mapsto x$, $x \in \mathcal{M}$. In what follows we shall identify $\mathrm{End}({}_{\mathcal{A}}\{\mathcal{M}_d/\mathcal{L}(d)\})$ with $\mathcal{D}$ via

$$[d \otimes x + \mathcal{L}(d)]\lambda = d \otimes (x\lambda) + \mathcal{L}(d),$$

$x \in \mathcal{M}$, $\lambda \in \mathcal{D}$.

The proof of the following result is an easy modification of that of [23, Proposition 2, p. 85] and is included for the sake of completeness.

Proposition 4.5: *Let $\mathcal{A}$ be a ring with simple left module $\mathcal{M}$ and let $d\colon \mathcal{A} \to \mathrm{End}(\mathcal{M}_{\mathcal{D}})$ be an $\mathcal{M}$-outer derivation. Then $\mathcal{M}_d$ is a local left $\mathcal{A}$-module with maximal submodule $\mathcal{L}(d)$.*

Proof: If $\mathcal{M}_d$ is local, then $\mathcal{L}(d)$ is a maximal submodule, because it is a proper submodule of $\mathcal{M}_d$ and $\mathcal{M}_d/\mathcal{L}(d)$ is simple.

Suppose that $\mathcal{M}_d$ is not local. Then there exists a proper $\mathcal{A}$-submodule $\mathcal{N}$ of $\mathcal{M}_d$ which is not contained in $\mathcal{L}(d)$. Since $\mathcal{L}(d)$ is a simple $\mathcal{A}$-module, we have that either $\mathcal{N} \supset \mathcal{L}(d)$ or $\mathcal{N} \cap \mathcal{L}(d) = \{0\}$. If $\mathcal{N} \supset \mathcal{L}(d)$, then $\mathcal{N}/\mathcal{L}(d)$ must be a nonzero proper submodule of the simple $\mathcal{A}$-module $\mathcal{M}_d/\mathcal{L}(d) \cong \mathcal{M}$, which is impossible. Therefore $\mathcal{N} \cap \mathcal{L}(d) = \{0\}$. It follows that $\mathcal{N}$ is isomorphic to the simple left $\mathcal{A}$-module $\mathcal{M}_d/\mathcal{L}(d) \cong \mathcal{M}$. Let $0 \neq e \otimes y + d \otimes x \in \mathcal{N}$. Then $x \neq 0$

since $\mathcal{N} \cap \mathcal{L}(d) = \{0\}$. It is clear that y is uniquely determined by x because $\mathcal{N} \cap \mathcal{L}(d) = \{0\}$. Further,

$$a(e \otimes y + d \otimes x) = e \otimes (ay + a^d x) + d \otimes ax, \quad a \in \mathcal{A}.$$

As $x \neq 0$, $\mathcal{A}x = \mathcal{M}$ and so for every $u \in \mathcal{M}$ there exists a uniquely determined $v \in \mathcal{M}$ with $e \otimes v + d \otimes u \in \mathcal{N}$. We define a map $T: \mathcal{M} \to \mathcal{M}$ by the rule $Tu = v$. Clearly T is well-defined and $T \in \mathrm{End}(\mathcal{M})$. Since $a(e \otimes v + d \otimes u) = e \otimes (av + a^d u) + d \otimes au$, we see that $T(au) = av + a^d u = aTu + a^d u$ and so $[T, L_a]u = a^d u$ for all $u \in \mathcal{M}$. That is, $[T, L_a] = a^d$ for all $a \in \mathcal{A}$, a contradiction. ∎

The next result generalizes Theorem 1.1.

THEOREM 4.6: *Let $\mathcal{A}$ be a ring with simple module ${}_{\mathcal{A}}\mathcal{M}$, let $\mathcal{D} = \mathrm{End}({}_{\mathcal{A}}\mathcal{M})$ and let d: $\mathcal{A} \to \mathrm{End}(\mathcal{M}_{\mathcal{D}})$ be a derivation. Then the following conditions are equivalent:*

(1) *d is $\mathcal{M}$-outer;*
(2) *given any elements $x_1, x_2, \ldots, x_n \in \mathcal{M}$ linearly independent over $\mathcal{D}$ and arbitrary elements $y_1, y_2, \ldots, y_n, z_1, z_2, \ldots, z_n \in \mathcal{M}$, there exists an element $a \in \mathcal{A}$ such that*

$$ax_i = y_i, \quad a^d x_i = z_i, \quad i = 1, 2, \ldots, n.$$

Proof: Suppose that d is $\mathcal{M}$-outer. Then $\mathcal{M}_d$ is a local $\mathcal{A}$-module. Set $\overline{x}_i = d \otimes x_i$ and $\overline{y}_i = d \otimes y_i + e \otimes z_i$, $i = 1, 2, \ldots, n$. By Theorem 2.3, there exists $a \in \mathcal{A}$ such that $a\overline{x}_i = \overline{y}_i$ for all $i = 1, 2, \ldots, n$. That is to say, $ax_i = y_i$ and $a^d x_i = z_i$ for all $i = 1, 2, \ldots, n$.

Conversely, suppose that (2) is fulfilled and there exist $T \in \mathrm{End}(\mathcal{M}_{\mathcal{D}})$ with $a^d = [T, L_a]$ for all $a \in \mathcal{A}$. We shall argue as in [10]. Pick $0 \neq x \in \mathcal{M}$. If x and Tx are linearly independent over $\mathcal{D}$, then by the assumption there exists $a \in \mathcal{A}$ with $ax = 0 = aTx$ and $a^d x = x$. If x and Tx are linearly dependent, we choose $a \in \mathcal{A}$ such that $ax = 0$ and $a^d x = x$. In this case we also have that $aTx = 0$. We now have that

$$x = a^d x = [T, L_a]x = Tax - aTx = 0,$$

a contradiction. Thus d is $\mathcal{M}$-outer. ∎

We note that, analogously to the automorphism case, Theorem 4.2 can be deduced at once from Theorem 4.6. Basically we will just repeat the argument given in [47, Remark 3.5]. Let $\mathcal{A}$ be a primitive ring with nonzero socle, d: $\mathcal{A} \to$

$\mathrm{End}(\mathcal{M}_{\mathcal{D}})$ be a derivation, and $\mathcal{M}$ be a faithful simple left $\mathcal{A}$-module. Pick a nonzero $a \in \mathcal{A}$ such that $ax_0 = x_0$ and $a\mathcal{M} = x_0\mathcal{D}$ for some nonzero $x_0 \in \mathcal{M}$. Given any $b \in \mathcal{A}$ such that $bx_0 = b(a^d x_0) = 0$, we have $ba = 0$ and hence $b^d a + ba^d = 0$. In particular, $b^d x_0 = 0$. But then d is $\mathcal{M}$-inner by Theorem 4.6.

The goal of this section is to prove an analogue of Theorem 4.6 for more derivations. Consider derivations d_i: $\mathcal{A} \to \mathrm{End}(\mathcal{M}_{\mathcal{D}})$, $i = 1, \ldots, n$. Denoting by $D_{\mathcal{M}}(\mathcal{A})$ the set of all $\mathcal{M}$-inner derivations, we shall say that $d_1, d_2, \ldots, d_n$ are **dependent over $\mathcal{D}$ modulo $D_{\mathcal{M}}(\mathcal{A})$**, if there exist elements $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathcal{D}$ not all 0 and $T \in \mathrm{End}(\mathcal{M})$ such that $\sum_{i=1}^n a^{d_i} x\lambda_i + [T, L_a]x = 0$ for all $a \in \mathcal{A}$ and $x \in \mathcal{M}$; otherwise they are called independent over $\mathcal{D}$ modulo $D_{\mathcal{M}}(\mathcal{A})$. Let V be a vector space over F with basis $\{\overline{d}_n, e_1, e_2, \ldots, e_n\}$. We set $\mathcal{M}_{\overline{d}_n} = V \otimes_F \mathcal{M}$. Clearly $\mathcal{M}_{\overline{d}_n}$ is a left $\mathcal{A}$-module under the multiplication

$$a\left(\overline{d}_n \otimes x + \sum_{i=1}^n e_i \otimes x_i\right) = \overline{d}_n \otimes ax + \sum_{i=1}^n e_i \otimes (a^{d_i} x + ax_i)$$

for all $a \in \mathcal{A}$, $x, x_1, x_2, \ldots, x_n \in \mathcal{M}$. We set $\mathcal{L}(\overline{d}_n) = \sum_{i=1}^n e_i \otimes \mathcal{M}$. Clearly $\mathcal{L}(\overline{d}_n)$ is a submodule of $\mathcal{M}_{\overline{d}_n}$ isomorphic to the direct sum of n copies of $\mathcal{M}$. Obviously $\mathcal{M}_{\overline{d}_n}/\mathcal{L}(\overline{d}_n) \cong \mathcal{M}$. The reader will see that the modules $\mathcal{M}_{\overline{d}_n}$ and $\mathcal{L}(\overline{d}_n)$ will play the same role in proving the density theorem for derivations $d_1, d_2, \ldots, d_n$ as modules $\mathcal{M}_d$ and $\mathcal{L}(d)$ in the proof of Theorem 4.6. We first prove the following generalization of Proposition 4.5.

PROPOSITION 4.7: *Suppose that $d_1, d_2, \ldots, d_n$ are independent over $\mathcal{D}$ modulo $D_{\mathcal{M}}(\mathcal{A})$. Then $\mathcal{M}_{\overline{d}_n}$ is a local module with maximal submodule $\mathcal{L}(\overline{d}_n)$.*

Proof: We proceed by induction on n. The case $n = 1$ follows from Proposition 4.5. In the inductive case we assume that $\mathcal{M}_{\overline{d}_{n-1}}$ is a local module with maximal submodule $\mathcal{L}(\overline{d}_{n-1}) = \sum_{i=1}^{n-1} e_i \otimes \mathcal{M}$.

Pick $x, x_1, x_2, \ldots, x_n \in \mathcal{M}$ with $x \neq 0$ and set

$$\overline{x} = \overline{d}_{n-1} \otimes x + \sum_{i=1}^{n-1} e_i \otimes x_i \in \mathcal{M}_{\overline{d}_{n-1}} \text{ and } \overline{y} = d_n \otimes x + e \otimes x_n \in \mathcal{M}_{d_n}.$$

Suppose that $a\overline{x} = 0$, $a \in \mathcal{A}$, implies $a\overline{y} = 0$. Note that $\mathcal{A}\overline{x} = \mathcal{M}_{\overline{d}_{n-1}}$ and $\mathcal{A}\overline{y} = \mathcal{M}_{d_n}$ by Remark 2.2(4). It follows that there exists an epimorphism of modules β: $\mathcal{M}_{\overline{d}_{n-1}} \to \mathcal{M}_{d_n}$. By Remark 2.2(1), β maps the Jacobson radical $\mathcal{L}(\overline{d}_{n-1})$ of $\mathcal{M}_{\overline{d}_{n-1}}$ into the Jacobson radical $\mathcal{L}(d_n) = e \otimes \mathcal{M}$ of $\mathcal{M}_{d_n}$. Since each

$e_i \otimes \mathcal{M} \cong \mathcal{M}$, there exist elements $\lambda_1, \lambda_2, \ldots, \lambda_{n-1} \in \mathcal{D}$ such that

$$\left(\sum_{i=1}^{n-1} e_i \otimes z_i\right)\beta = \sum_{i=1}^{n-1} e \otimes z_i\lambda_i \quad \text{for all } z_1, \ldots, z_n \in \mathcal{M}.$$

If $(\mathcal{L}(\overline{d}_{n-1}))\beta = 0$, then β induces an epimorphism of the simple module $\mathcal{M} \cong \mathcal{M}_{\overline{d}_{n-1}}/\mathcal{L}(\overline{d}_{n-1})$ onto the local module $\mathcal{M}_{d_n}$, which is impossible. Therefore $(\mathcal{L}(\overline{d}_{n-1}))\beta \neq 0$ and so not all λ_i's are equal to 0.

Clearly β induces a homomorphism of modules

$$\mathcal{M} \cong \mathcal{M}_{\overline{d}_{n-1}}/\mathcal{L}(\overline{d}_{n-1}) \to \mathcal{M}_{d_n}/\mathcal{L}(d_n) \cong \mathcal{M}$$

and so there exist elements $\lambda \in \mathcal{D}$ and $T \in \mathrm{End}(\mathcal{M})$ such that

$$(\overline{d}_{n-1} \otimes z)\beta = d_n \otimes z\lambda + e \otimes Tz \quad \text{for all } z \in \mathcal{M}.$$

Given $a \in \mathcal{A}$ and $z \in \mathcal{M}$, we now have

$$\begin{aligned} d_n \otimes az\lambda + e \otimes a^{d_n} z\lambda + e \otimes aTz &= a(d_n \otimes z\lambda + e \otimes Tz) = a[(\overline{d}_{n-1} \otimes z)\beta] \\ &= [a(\overline{d}_{n-1} \otimes z)]\beta = \left(\overline{d}_{n-1} \otimes az + \sum_{i=1}^{n-1} e_i \otimes a^{d_i} z\right)\beta \\ &= d_n \otimes az\lambda + e \otimes T(az) + \sum_{i=1}^{n-1} e \otimes a^{d_i} z\lambda_i \end{aligned}$$

and hence

$$\sum_{i=1}^{n-1} a^{d_i} z\lambda_i - a^{d_n} z\lambda + [T, L_a]z = 0$$

for all $a \in \mathcal{A}$ and $z \in \mathcal{M}$, a contradiction. This shows that there exists $a \in \mathcal{A}$ such that $a\overline{x} = 0$ and $a\overline{y} \neq 0$. That is,

$$(6) \qquad ax = a^{d_1}x + ax_1 = \cdots = a^{d_{n-1}}x + ax_{n-1} = 0 \quad \text{and} \quad a^{d_n}x + ax_n \neq 0.$$

Let $\overline{z} \in \mathcal{M}_{\overline{d}_n} \setminus \mathcal{L}(\overline{d}_n)$. According to Remark 2.2(4), it is enough to show that $\mathcal{A}\overline{z} = \mathcal{M}_{\overline{d}_n}$. To this end, write $\overline{z} = \overline{d}_n \otimes x + \sum_{i=1}^{n} e_i \otimes x_i$ where $x, x_i \in \mathcal{M}$. Clearly $x \neq 0$. Therefore, by what we have just shown, there exists $a \in \mathcal{A}$ such that (6) is fulfilled. Hence $a\overline{z} = e_n \otimes (a^{d_n}x + ax_n) \neq 0$ and so $e_n \otimes \mathcal{M} \subseteq \mathcal{A}\overline{z}$. Since $\mathcal{M}_{\overline{d}_n}/(e_n \otimes \mathcal{M}) \cong \mathcal{M}_{\overline{d}_{n-1}}$ which is local by the induction assumption, we conclude that

$$\mathcal{A}\overline{z} = \mathcal{A}\overline{z} + e_n \otimes \mathcal{M} = \mathcal{M}_{\overline{d}_n}. \qquad \blacksquare$$

We are now in a position to prove the main result of this section.

THEOREM 4.8: *Let $\mathcal{A}$ be a ring with simple module ${}_{\mathcal{A}}\mathcal{M}$, let $\mathcal{D} = \mathrm{End}({}_{\mathcal{A}}\mathcal{M})$ and let d_i: $\mathcal{A} \to \mathrm{End}(\mathcal{M}_{\mathcal{D}})$, $i = 1, 2, \dots, m$, be derivations. Then the following conditions are equivalent:*

(1) *$d_1, d_2, \dots, d_m$ are independent over $\mathcal{D}$ modulo $D_{\mathcal{M}}(\mathcal{A})$;*

(2) *given any elements $x_1, x_2, \dots, x_n \in \mathcal{M}$ linearly independent over $\mathcal{D}$ and arbitrary elements $y_i, z_{ij} \in \mathcal{M}$, $1 \le i \le n$, $1 \le j \le m$, there exists an element $a \in \mathcal{A}$ such that*

$$ax_i = y_i, \quad a^{d_j} x_i = z_{ij}, \quad i = 1, 2, \dots, n, \; j = 1, 2, \dots, m.$$

Proof: Suppose that (1) is fulfilled. Then $\mathcal{M}_{\overline{d}_m}$ is a local $\mathcal{A}$-module by Proposition 4.7. Set $\overline{x}_i = \overline{d}_m \otimes x_i$ and $\overline{y}_i = \overline{d}_m \otimes y_i + \sum_{j=1}^{m} e_j \otimes z_{ij}$, $i = 1, 2, \dots, n$. By Theorem 2.3 there exists $a \in \mathcal{A}$ such that $a\overline{x}_i = \overline{y}_i$ for all $i = 1, 2, \dots, n$. Clearly a is the desired element.

Now suppose that (1) does not hold, that is, that there exist elements $\lambda_i \in \mathcal{D}$, $1 \le i \le n$, with $\lambda_1 \ne 0$ and $T \in \mathrm{End}(\mathcal{M})$ such that $\sum_{i=1}^{n} a^{d_i} x \lambda_i + [T, L_a]x = 0$ for all $a \in \mathcal{A}$ and $x \in \mathcal{M}$. Assume that (2) holds true. Arguing similarly as in the proof of Theorem 4.6 we see that given any nonzero $x \in \mathcal{M}$ we can find $a \in \mathcal{A}$ such that $ax = aTx = a^{d_2}x = \dots = a^{d_m}x = 0$ and $a^{d_1}x = x$, which yields $x\lambda_1 = 0$. With this contradiction the theorem is proved. ∎

5. The main theorem

Let $\mathcal{A} \subseteq \mathcal{B}$ be rings, let $\mathcal{M}$ be a left $\mathcal{B}$-module which is simple as an $\mathcal{A}$-module, and assume that $\mathcal{D} = \mathrm{End}({}_{\mathcal{A}}\mathcal{M})$ is equal to $\mathrm{End}({}_{\mathcal{B}}\mathcal{M})$ (the main reason for dealing with two rings is that we shall need this in the next section). Let $D(\mathcal{B})$ be the additive group of derivations of $\mathcal{B} \to \mathcal{B}$. Given $d \in D(\mathcal{B})$, it induces a derivation $\overline{d}$: $\mathcal{A} \to \mathrm{End}(\mathcal{M}_{\mathcal{D}})$ given by the rule $a^{\overline{d}} = L_{a^d}$. Let n be a positive integer. We shall say that derivations $d_1, d_2, \dots, d_n \in D(\mathcal{B})$ are **independent over $\mathcal{D}$ modulo** $D_{\mathcal{M}}(\mathcal{A})$ if $\overline{d}_1, \overline{d}_2, \dots, \overline{d}_n$ are independent over $\mathcal{D}$ modulo $D_{\mathcal{M}}(\mathcal{A})$. In what follows F is the prime subfield of $\mathcal{D}$.

Let $n, m_1, m_2, \dots, m_n$ be positive integers and let $d_1, d_2, \dots, d_n \in D(\mathcal{B})$. We set

$$\begin{aligned} \overline{m} &= (m_1, m_2, \dots, m_n), \\ \Omega(\overline{m}) &= \{\overline{s} = (s_1, s_2, \dots, s_n) \colon 0 \le s_i \le m_i, i = 1, 2, \dots, n\}, \\ \Delta_{\overline{s}} &= d_1^{s_1} d_2^{s_2} \dots d_n^{s_n}, \quad \overline{s} \in \Omega(\overline{m}) \quad \text{and} \\ \overline{0} &= (0, 0, \dots, 0). \end{aligned}$$

It is understood that $\Delta_{\overline{0}} = e$, the identity map $\mathcal{B} \to \mathcal{B}$. Given $\overline{s}, \overline{r} \in \Omega(\overline{m})$, we shall write $\overline{s} \ge \overline{r}$ provided that $s_i \ge r_i$ for all $i = 1, 2, \dots, n$. Let $a, b \in \mathcal{A}$ and

$\overline{s} \in \Omega(\overline{m})$. According to Leibnitz Formula [4, Remark 1.1.1],

$$(ab)^{\Delta_{\overline{s}}} = \sum_{\substack{\overline{r}\in\Omega(\overline{m}),\\ \overline{r}\leq\overline{s}}} \binom{\overline{s}}{\overline{r}} a^{\Delta_{\overline{s}-\overline{r}}} b^{\Delta_{\overline{r}}} \tag{7}$$

where $\binom{\overline{s}}{\overline{r}} = \prod_{i=1}^{n} \binom{s_i}{r_i}$. Let V be the vector space over F with basis $\{\Delta_{\overline{s}}\colon \overline{s} \in \Omega(\overline{m})\}$. We set $\mathcal{M}_{\Omega(\overline{m})} = V \otimes_F \mathcal{M}$. When the context is clear, we shall simply write $\mathcal{M}_\Omega$ for $\mathcal{M}_{\Omega(\overline{m})}$ and Ω for $\Omega(\overline{m})$. It follows from (7) that $\mathcal{M}_\Omega$ is a left $\mathcal{A}$-module under the operation

$$a(\Delta_{\overline{s}} \otimes x) = \sum_{\overline{r}\in\Omega,\overline{r}\leq\overline{s}} \binom{\overline{s}}{\overline{r}} \Delta_{\overline{s}-\overline{r}} \otimes a^{\Delta_{\overline{r}}} x \tag{8}$$

for all $a \in \mathcal{A}$, $x \in \mathcal{M}$, $\overline{s} \in \Omega$. Moreover,

$$\mathcal{L}(\Omega) = \sum_{\overline{s}\in\Omega,\overline{s}<\overline{m}} \Delta_{\overline{s}} \otimes \mathcal{M}$$

is a submodule of $\mathcal{M}_\Omega$ such that

$$\mathcal{M}_\Omega/\mathcal{L}(\Omega) \cong \mathcal{M} \tag{9}$$

as $\mathcal{A}$-modules. Given $\overline{s} \in \Omega$, note that

$$\mathcal{M}_{\Omega(\overline{s})} = \sum_{\overline{r}\in\Omega(\overline{s})} \Delta_{\overline{r}} \otimes \mathcal{M} \subseteq \mathcal{M}_\Omega \tag{10}$$

for all $\overline{s} \in \Omega$. Finally, if $\overline{s} = (s_1, s_2, \ldots, s_n) \in \Omega$, we set $|\overline{s}| = \sum_{i=1}^{n} s_i$. The reader will see that the modules $\mathcal{M}_\Omega$ and $\mathcal{L}(\Omega)$ are the main tool in the proof of our main result.

PROPOSITION 5.1: *Suppose that the following conditions are fulfilled:*
(1) *$d_1, d_2, \ldots, d_n \in D(\mathcal{B})$ are independent over $\mathcal{D}$ modulo $D_{\mathcal{M}}(\mathcal{A})$;*
(2) *either $\mathrm{char}(D) = 0$ or $\mathrm{char}(D) = p > 0$ and each $m_i < p$.*
Then $\mathcal{M}_\Omega$ is a local $\mathcal{A}$-module with maximal submodule $\mathcal{L}(\Omega)$.

Proof: Given $\overline{x} \in \mathcal{M}_\Omega \setminus \mathcal{L}(\Omega)$, in view of Remark 2.2(4) it is enough to show that $\mathcal{A}\overline{x} = \mathcal{M}_\Omega$. In order to prove the equality, we shall make use of the triple induction. The first induction is on $|\overline{m}|$. On the inductive step on $|\overline{m}|$ we shall represent $\mathcal{L}(\Omega)$ as a union of certain $\mathcal{A}$-submodules $\mathcal{N}_k$, $-1 \leq k \leq |\overline{m}| - 1$ (with $\mathcal{N}_{-1} = 0$) and proceed by induction on k to show that $\mathcal{N}_k \subseteq \mathcal{A}\overline{x}$ for all k (and so $\mathcal{L}(\Omega) \subset \mathcal{A}\overline{x}$ forcing $\mathcal{A}\overline{x} = \mathcal{M}_\Omega$ in view of (9)). Making the induction step on k, we shall introduce the concept of the height $h(z)$ of the element $z \in \mathcal{A}\overline{x} \setminus \mathcal{N}_{k-1}$ and proceed by the induction on $h(z)$.

We now proceed by induction on $|\overline{m}|$. If $|\overline{m}| = 1$, then $n = 1 = m_1$. In this case $\mathcal{M}_\Omega = \mathcal{M}_{d_1}$, $\mathcal{L}(\Omega) = \mathcal{L}(d_1)$ and the result follows from Proposition 4.5.

In the inductive case we may assume that each $\mathcal{M}_{\Omega(\overline{s})}$ is local for all $\overline{s} \in \Omega$ with $|\overline{s}| < |\overline{m}|$. Let

$$\overline{x} = \sum_{\overline{s}\in\Omega} \Delta_{\overline{s}} \otimes x_{\overline{s}} \in \mathcal{M}_\Omega$$

with $x_{\overline{m}} \neq 0$. By Remark 2.2(4) it is enough to show that $\mathcal{A}\overline{x} = \mathcal{M}_\Omega$. It follows from (8) that

$$a\overline{x} - \Delta_{\overline{m}} \otimes ax_{\overline{m}} \in \mathcal{L}(\Omega)$$

and so $\mathcal{A}\overline{x} + \mathcal{L}(\Omega) = \Delta_{\overline{m}} \otimes \mathcal{A}x_{\overline{m}} + \mathcal{L}(\Omega) = \mathcal{M}_\Omega$. Therefore it is enough to show that $\mathcal{A}\overline{x} \supseteq \mathcal{L}(\Omega)$.

Let k be a nonnegative integer. We set $\mathcal{N}_{-1} = 0$ and

$$\mathcal{N}_k = \sum_{\overline{s}\in\Omega, |\overline{s}|\leq k} \Delta_{\overline{s}} \otimes \mathcal{M} \subseteq \mathcal{M}_\Omega.$$

Clearly $\mathcal{L}(\Omega) = \mathcal{N}_{|\overline{m}|-1}$. We proceed by induction on k to prove that $\mathcal{N}_k \subseteq \mathcal{A}\overline{x}$. The case $k = -1$ is clear. In the inductive case we assume that $\mathcal{A}\overline{x} \supseteq \mathcal{N}_{k-1}$. If $k = |\overline{m}|$, then $\mathcal{N}_{k-1} = \mathcal{L}(\Omega)$ and there is nothing to prove. Therefore we may assume that $k < |\overline{m}|$. Pick any $\overline{s} \in \Omega$ with $|\overline{s}| = k$. Since

$$\mathcal{N}_k = \sum_{\overline{r}\in\Omega, |\overline{r}|=k} \Delta_{\overline{r}} \otimes \mathcal{M} + \mathcal{N}_{k-1},$$

it suffices to show that $\Delta_{\overline{s}} \otimes \mathcal{M} \subseteq \mathcal{A}\overline{x}$. To this end, we introduce the following concept. A nonzero element $\overline{y} = \sum_{\overline{s}\in\Omega} \Delta_{\overline{s}} \otimes y_{\overline{s}} \in \mathcal{M}_\Omega$ is said to have a height provided that there exists $\overline{r} \in \Omega$ such that

$$\overline{y} = \Delta_{\overline{r}} \otimes y_{\overline{r}} + \sum_{\overline{p}\in\Omega, |\overline{p}|<|\overline{r}|} \Delta_{\overline{p}} \otimes y_{\overline{p}} \quad \text{and} \quad y_{\overline{r}} \neq 0.$$

In this case we shall write $h(\overline{y}) = \overline{r}$. For example, $h(\overline{x}) = \overline{m}$. In the set

$$\{\overline{y} \in \mathcal{A}\overline{x} \setminus \mathcal{N}_{k-1} \colon h(\overline{y}) \geq \overline{s}\}$$

we choose an element $\overline{z}$ with minimal possible height (with respect to the partial order on Ω). Suppose that $|h(\overline{z})| = k$. Then $\overline{z} = \Delta_{\overline{s}} \otimes z_{\overline{s}} + \overline{u}$ for some $\overline{u} \in \mathcal{N}_{k-1}$. Therefore

$$\mathcal{A}\overline{x} \supseteq \mathcal{A}\overline{z} + \mathcal{N}_{k-1} \supseteq \mathcal{A}(\Delta_{\overline{s}} \otimes z_{\overline{s}}) = \mathcal{M}_{\Omega(\overline{s})}$$

because $\mathcal{M}_{\Omega(\overline{s})}$ is local by the induction assumption (recall that $k < |\overline{m}|$). Therefore we may assume that $|h(\overline{z})| > k = |\overline{s}|$. Let $\overline{r} = h(\overline{z})$. We have $\overline{r} > \overline{s}$ and whence there exists index i such that $r_i > s_i$. Say, $i = 1$. By Theorem 4.8 there exists $a \in \mathcal{A}$ such that

$$\begin{aligned} az_{\overline{p}} &= 0, \quad \overline{p} \in \Omega, \\ a^{d_i} z_{\overline{r}} &= 0, \quad i = 2, 3, \ldots, n, \quad \text{and} \\ a^{d_1} z_{\overline{r}} &\neq 0. \end{aligned}$$

Therefore

$$a\overline{z} = r_1 \Delta_{\overline{q}} \otimes a^{d_1} z_{\overline{r}} + \sum_{\substack{\overline{p} \in \Omega, \\ |\overline{p}| < |\overline{r}| - 1}} \Delta_{\overline{p}} \otimes w_{\overline{p}}$$

for suitable $w_{\overline{p}} \in \mathcal{M}$, where $\overline{q} = (r_1 - 1, r_2, \ldots, r_n)$. By our assumption r_1 is a nonzero element of F and whence the element $a\overline{z}$ has a height and $\overline{r} > h(a\overline{z}) = \overline{q} \geq \overline{s}$, a contradiction. ∎

THEOREM 5.2: *Let $\mathcal{A} \subseteq \mathcal{B}$ be rings, let k be a positive integer, let $\mathcal{M}_i$, $i = 1, 2, \ldots, k$, be left $\mathcal{B}$-modules which are simple as $\mathcal{A}$-modules. Further, let $l, n, m_1, m_2, \ldots, m_n$ be positive integers, let $d_1, d_2, \ldots, d_n \in D(\mathcal{B})$, let $x_{i1}, x_{i2}, \ldots, x_{il} \in \mathcal{M}_i$ be elements linearly independent over $\mathcal{D}_i = \mathrm{End}({}_{\mathcal{A}}\mathcal{M}_i)$ and let $y_{ij}, z_{ij\overline{s}} \in \mathcal{M}_i$, $i = 1, 2, \ldots, k$, $j = 1, 2, \ldots, l$, $\overline{s} \in \Omega = \Omega(\overline{m})$, $\overline{s} \neq \overline{0}$. Suppose that $\mathrm{End}({}_{\mathcal{A}}\mathcal{M}_i) = \mathrm{End}({}_{\mathcal{B}}\mathcal{M}_i)$ for all $i = 1, 2, \ldots, k$ and the following conditions are fulfilled:*

(1) *$d_1, d_2, \ldots, d_n$ are independent over $\mathcal{D}_i$ modulo $D_{\mathcal{M}_i}(\mathcal{A})$ for all $i = 1, \ldots, k$;*
(2) *for all $i = 1, 2, \ldots, k$, either $\mathrm{char}(\mathcal{D}_i) = 0$ or $\mathrm{char}(\mathcal{D}_i) = p_i > 0$ and each $m_j < p_i$;*
(3) *${}_{\mathcal{A}}\mathcal{M}_i \not\cong {}_{\mathcal{A}}\mathcal{M}_j$ for all $i \neq j$.*

Then there exists $a \in \mathcal{A}$ such that

$$ax_{ij} = y_{ij}, \quad a^{\Delta_{\overline{s}}} x_{ij} = z_{ij\overline{s}}$$

for all $i = 1, 2, \ldots, k$, $j = 1, 2, \ldots, l$, $\overline{0} \neq \overline{s} \in \Omega$.

Proof: By Proposition 5.1, each $\mathcal{N}_i = (\mathcal{M}_i)_\Omega$ is a local $\mathcal{A}$-module with maximal proper submodule $\mathcal{L}_i$ such that $\mathcal{N}_i/\mathcal{L}_i \cong \mathcal{M}_i$. It follows from (3) that $\mathcal{N}_i/\mathcal{L}_i \not\cong \mathcal{N}_j/\mathcal{L}_j$ if $i \neq j$. Set $\overline{x}_{ij} = \Delta_{\overline{m}} \otimes x_{ij}$ and

$$\overline{y}_{ij} = \Delta_{\overline{m}} \otimes y_{ij} + \sum_{\overline{s} \in \Omega, \overline{s} \neq \overline{0}} \binom{\overline{m}}{\overline{s}} \Delta_{\overline{m} - \overline{s}} \otimes z_{ij\overline{s}}, \quad i = 1, 2, \ldots, k, \ j = 1, 2, \ldots, l.$$

Note that each $\binom{\overline{m}}{\overline{s}}$ is a nonzero element of $\mathcal{D}_i$ for all $i = 1, 2, \ldots, k$. By Theorem 2.3 there exists $a \in \mathcal{A}$ such that $a\overline{x}_{ij} = \overline{y}_{ij}$ for all $i = 1, 2, \ldots, k$, $j = 1, 2, \ldots, l$. The result now follows from (8). ∎

Finally, we have arrived at the main result of the present article.

THEOREM 5.3: *Let $\mathcal{A} \subseteq \mathcal{B}$ be rings. Further, let $\mathcal{M}$ be a left $\mathcal{B}$-module which is simple as an $\mathcal{A}$-module and such that* $\mathrm{End}({}_{\mathcal{A}}\mathcal{M}) = \mathrm{End}({}_{\mathcal{B}}\mathcal{M})$. *Let $k, l, n, m_1, m_2, \ldots, m_n$ be positive integers, let $d_1, \ldots, d_n \in D(\mathcal{B})$ and let $\alpha_1, \ldots, \alpha_l$ be automorphisms of the ring $\mathcal{A}$. Suppose that the following conditions are fulfilled:*

(1) *$d_1, d_2, \ldots, d_n$ are independent over $\mathcal{D} = \mathrm{End}({}_{\mathcal{A}}\mathcal{M})$ modulo $D_{\mathcal{M}}(\mathcal{A})$;*
(2) *either $\mathrm{char}(D) = 0$ or $\mathrm{char}(D) = p > 0$ and each $m_i < p$;*
(3) *α_i and α_j are $\mathcal{M}$-independent for all $i \neq j$.*

Then for any elements $x_1, x_2, \ldots, x_k \in \mathcal{M}$ linearly independent over $\mathcal{D}$ and for any $z_{i\overline{s}j} \in \mathcal{M}$, $i = 1, 2, \ldots, k$, $\overline{s} \in \Omega = \Omega(\overline{m})$, $j = 1, 2, \ldots, l$, there exists $a \in \mathcal{A}$ with

$$a^{\Delta_{\overline{s}}\alpha_j} x_i = z_{i\overline{s}j} \quad \textit{for all } i = 1, 2, \ldots, k,\ \overline{s} \in \Omega,\ j = 1, 2, \ldots, l.$$

Proof: Set $\mathcal{N} = \mathcal{M}_\Omega$ and $\mathcal{L} = \mathcal{L}(\Omega)$. By Proposition 5.1, $\mathcal{N}$ is a local module with maximal submodule $\mathcal{L}$ and $\mathcal{N}/\mathcal{L} \cong \mathcal{M}$. Set $\mathcal{N}_j = \mathcal{N}_{\alpha_j}$ and $\mathcal{L}_j = \mathcal{L}_{\alpha_j} \subseteq \mathcal{N}_j$. It is easy to see that $\mathcal{N}_j$ is a local module with maximal submodule $\mathcal{L}_j$ and $\mathcal{N}_j/\mathcal{L}_j \cong \mathcal{M}_{\alpha_j}$. By Proposition 3.5, $\mathcal{N}_i/\mathcal{L}_i \cong \mathcal{N}_j/\mathcal{L}_j$ if and only if $i = j$. According to (3), $\mathrm{End}({}_{\mathcal{A}}\mathcal{M}_{\alpha_j}) = \mathcal{D}$. In particular,

$$\Delta_{\overline{m}} \otimes x_1, \Delta_{\overline{m}} \otimes x_2, \ldots, \Delta_{\overline{m}} \otimes x_k$$

are independent elements of $\mathcal{N}_j$ over $\mathcal{D}$ modulo $\mathcal{L}_j$, $j = 1, 2, \ldots, l$. The result now follows from Theorem 5.2. ∎

6. The case of primitive rings

Throughout this section $\mathcal{A}$ will be a primitive ring with faithful simple left module $\mathcal{M}$, with extended centroid $\mathcal{C}$ and with central closure $\mathcal{A}_c$ (see [4, Chapter 2] or [32]). Let $\mathcal{D} = \mathrm{End}({}_{\mathcal{A}}\mathcal{M})$. According to [4, Theorem 4.1.1], we may assume that $\mathcal{A}_c$ is a subring of $\mathrm{End}(\mathcal{M}_{\mathcal{D}})$ and the action of $\mathcal{A}_c$ on $\mathcal{M}$ extends that of $\mathcal{A}$. In particular $\mathcal{M}$ is a faithful simple $\mathcal{A}_c$-module and $\mathrm{End}({}_{\mathcal{A}_c}\mathcal{M}) = \mathcal{D}$. Therefore we may assume that $\mathcal{C}$ is contained in the center of $\mathcal{D}$. In what follows F is the prime subfield of both $\mathcal{C}$ and $\mathcal{D}$.

Recall that elements $t_1, t_2, \ldots, t_n \in \mathrm{End}(\mathcal{M})$ are said to be linearly dependent over $\mathcal{D}$ if there exist $\lambda_i \in \mathcal{D}$, $i = 1, 2, \ldots, n$, not all 0, such that $\sum_{i=1}^{n}(t_i x)\lambda_i = 0$ for all $x \in \mathcal{M}$.

LEMMA 6.1 ([4, Corollary 4.2.5]): *Elements $a_1, a_2, \ldots, a_n \in \mathcal{A}_c$ are linearly dependent over $\mathcal{C}$ if and only if they are linearly dependent over $\mathcal{D}$.*

Let $d\colon \mathcal{A} \to \mathcal{A}$ be a derivation. By [4, Proposition 2.5.1], d can be extended to the uniquely determined derivation $d\colon Q_s \to Q_s$, where Q_s is the symmetric ring of quotients of $\mathcal{A}$. Recall that $\mathcal{C}$ is the center of Q_s and so $\mathcal{C}^d \subseteq \mathcal{C}$. Because $\mathcal{A}_c = \mathcal{A}\mathcal{C}$, we conclude that $\mathcal{A}_c^d \subseteq \mathcal{A}_c$. Therefore every derivation d of $\mathcal{A}$ induces the unique derivation $d\colon \mathcal{A}_c \to \mathcal{A}_c$. Note that the set $D(\mathcal{A}_c)$ of all derivations $\mathcal{A}_c \to \mathcal{A}_c$ is a right $\mathcal{C}$-space under the operation $a^{dc} = ca^d$ for all $a \in \mathcal{A}_c$, $c \in \mathcal{C}$, $d \in D(\mathcal{A}_c)$. We denote by $D_{\mathcal{M}}(\mathcal{A}_c)$ the set of all $\mathcal{M}$-inner derivations $\mathcal{A}_c \to \mathcal{A}_c$. Clearly it is a subspace of $D(\mathcal{A}_c)$. We now set

$$\mathrm{Der}(\mathcal{A}) = D(\mathcal{A})\mathcal{C} + D_{\mathcal{M}}(\mathcal{A}_c) \subseteq D(\mathcal{A}_c).$$

One can easily check that $\mathrm{Der}(\mathcal{A})$ is a subspace of $D(\mathcal{A}_c)$.

LEMMA 6.2: *Let $d \in \mathrm{Der}(\mathcal{A})$ be a nonzero derivation. Then:*

(1) *if $\dim_{\mathcal{C}}(\mathcal{A}^d\mathcal{C}) = n < \infty$, then $\mathcal{A}$ is a simple Artinian ring of dimension $\leq 4n^2$ over its center;*

(2) *if d is $\mathcal{M}$-outer, then $\dim_{\mathcal{C}}(\mathcal{A}^d\mathcal{C}) = \infty$.*

Proof: (1) Assume that $\dim_{\mathcal{C}}(\mathcal{A}^d\mathcal{C}) = n < \infty$. Set $V = \mathcal{A}^d\mathcal{C}$. Pick $a \in \mathcal{A}$ such that $a^d \neq 0$. Given $x \in \mathcal{A}$, we have $a^d x = (ax)^d - ax^d \in V + aV$ and so

$$\dim_{\mathcal{C}}(a^d\mathcal{A}_c) \leq \dim_{\mathcal{C}}(V) + \dim_{\mathcal{C}}(aV) \leq 2n.$$

However, $\mathcal{A}_c \subseteq \mathrm{End}({}_{\mathcal{C}}\{a^d\mathcal{A}_c\})$ and whence $\dim_{\mathcal{C}}(\mathcal{A}_c) \leq 4n^2$. In particular $\mathcal{A}_c$ (and so $\mathcal{A}$) is a PI ring. Recalling that $\mathcal{A}$ is a primitive ring, we obtain from Kaplansky's theorem on primitive PI rings that $\mathcal{A}$ is a simple Artinian ring and whence $\mathcal{A} = \mathcal{A}_c$. Thus $\dim_{\mathcal{C}}(\mathcal{A}) \leq 4n^2$.

(2) By Corollary 4.3, $\mathcal{A}$ is not a simple Artinian ring and so $\dim_{\mathcal{C}}(\mathcal{A}^d\mathcal{C}) = \infty$. ∎

Given a field K, a vector space V over K and an additive abelian group U, the additive group $\mathrm{Hom}(U, V)$ of all additive maps $t\colon U \to V$ is a vector space over K under the operation $x^{t\lambda} = \lambda(x^t)$ for all $x \in U$, $\lambda \in K$, $t \in \mathrm{Hom}(U, V)$. Recall that $t \in \mathrm{Hom}(U, V)$ is said to be of finite rank if $\dim_K(KU^t) < \infty$. We now need the following version of Amitsur's lemma which is a corollary to [4, Theorem 4.2.7].

LEMMA 6.3: *Let V be a vector space over a field K, let U be an additive abelian group, and let $t_1, t_2, \dots, t_n \in \mathrm{Hom}(U, V)$ be linearly independent over K. Then either the subspace $\sum_{i=1}^n t_i K$ contains a map of finite rank, or there exists $u \in U$ such that $u^{t_1}, u^{t_2}, \dots, u^{t_n}$ are linearly independent over K.*

THEOREM 6.4: *Let n be a positive integer and $d_1, d_2, \dots, d_n \in \mathrm{Der}(\mathcal{A})$. Then $d_1, d_2, \dots, d_n$ are linearly independent over $\mathcal{C}$ modulo $D_{\mathcal{M}}(\mathcal{A}_c)$ if and only if they are linearly independent over $\mathcal{D}$ modulo $D_{\mathcal{M}}(\mathcal{A}_c)$.*

Proof: Since $\mathcal{C}$ is a subfield of the center of $\mathcal{D}$, the "if" part is clear. Assume that $d_1, d_2, \dots, d_n$ are linearly independent over $\mathcal{C}$ modulo $D_{\mathcal{M}}(\mathcal{A}_c)$. Let $0 \neq d_{n+1} \in D_{\mathcal{M}}(\mathcal{A}_c)$. By assumption $\mathcal{A}$ has $\mathcal{M}$-outer derivations and whence it is not a simple Artinian ring by Corollary 4.3. It now follows from Lemma 6.2 that $\dim_{\mathcal{C}}(\mathcal{A}^d\mathcal{C}) = \infty$ for every nonzero derivation $d \in \sum_{i=1}^{n+1} d_i\mathcal{C}$. By Lemma 6.3 there exists $a \in \mathcal{A}$ such that $a^{d_1}, a^{d_2}, \dots, a^{d_{n+1}}$ are linearly independent over $\mathcal{C}$. According to Lemma 6.1 they are linearly independent over $\mathcal{D}$ and thus $d_1, d_2, \dots, d_{n+1}$ are linearly independent over $\mathcal{D}$ for all $0 \neq d_{n+1} \in D_{\mathcal{M}}(\mathcal{A}_c)$. ∎

The following result is a corollary to Theorems 6.4 and 5.3 since $\mathrm{Der}(\mathcal{A}) \subseteq D(\mathcal{A}_c)$.

THEOREM 6.5: *Let $\mathcal{A}$ be a primitive ring with faithful simple left module $\mathcal{M}$ and with extended centroid $\mathcal{C}$, and let $k, l, n, m_1, m_2, \dots, m_n$ be positive integers. Let $\mathcal{D} = \mathrm{End}({}_{\mathcal{A}}\mathcal{M})$, let $d_1, d_2, \dots, d_n \in \mathrm{Der}(\mathcal{A})$ and let $\alpha_1, \alpha_2, \dots, \alpha_l$ be automorphisms of the ring $\mathcal{A}$. Suppose that the following conditions are fulfilled:*

(1) *$d_1, d_2, \dots, d_n$ are independent over $\mathcal{C}$ modulo $D_{\mathcal{M}}(\mathcal{A}_c)$;*
(2) *either $\mathrm{char}(F) = 0$ or $\mathrm{char}(F) = p > 0$ and each $m_i < p$;*
(3) *α_i and α_j are $\mathcal{M}$-independent for all $i \neq j$.*

Then for any elements $x_1, x_2, \dots, x_k \in \mathcal{M}$ linearly independent over $\mathcal{D}$ and for any $z_{i\overline{s}j} \in \mathcal{M}$, $i = 1, 2, \dots, k$, $\overline{s} \in \Omega = \Omega(\overline{m})$, $j = 1, 2, \dots, l$, there exists $a \in \mathcal{A}$ with

$$a^{\Delta_{\overline{s}}\alpha_j} x_i = z_{i\overline{s}j} \quad \text{for all } i = 1, 2, \dots, k,\ \overline{s} \in \Omega,\ j = 1, 2, \dots, l.$$

7. Applications

In this section we give a few indications of how the results above can be used when studying the so-called generalized identities in rings. The theory of generalized identities considers, in particular, derivations and automorphisms of rings satisfying certain identities or relations of polynomial type (see [4] for a vast literature); here we mention Herstein and some of his students who made a lot of contributions in this context, and Kharchenko who found a more systematic and uniform approach. Our results provide an alternative approach.

We first state a simple lemma, which is undoubtedly known. Nevertheless, we give the proof since it is rather short.

LEMMA 7.1: *Let $\mathcal{M}$ be a right vector space over a division ring $\mathcal{D}$ such that $\dim(\mathcal{M}_\mathcal{D}) \geq 2$ and T be an additive endomorphism of $\mathcal{M}$ such that x and Tx are linearly dependent for every $x \in \mathcal{M}$. Then there exists $\lambda \in \mathcal{D}$ such that $Tx = x\lambda$ for all $x \in \mathcal{M}$.*

Proof: Write $Tx = x\lambda_x$ where $\lambda_x \in \mathcal{D}$. Fix $0 \neq y \in \mathcal{M}$. Pick $x \in \mathcal{M}$ such that x, y are independent. We have $T(x+y) = (x+y)\lambda_{x+y}$ and, on the other hand, $T(x+y) = Tx + Ty = x\lambda_x + y\lambda_y$. Hence it follows that $\lambda_x = \lambda_{x+y} = \lambda_y$. Now let z be any nonzero element in $\mathcal{M}$. If y, z are independent, then $\lambda_z = \lambda_y$ by what we have just proved. If y, z are dependent, then x, z are independent and so $\lambda_z = \lambda_x = \lambda_y$. Thus $\lambda_z = \lambda_y$ for all $0 \neq z \in M$, which proves the lemma. ∎

We have to introduce some further notation. Let $\mathcal{A}$ be a ring and n be a positive integer. By $J_n(\mathcal{A})$ we denote the ideal of $\mathcal{A}$ consisting of those elements $a \in \mathcal{A}$ that $a\mathcal{M} = 0$ for every simple left $\mathcal{A}$-module $\mathcal{M}$ such that $\dim(\mathcal{M}_\mathcal{D}) \geq n$ where $\mathcal{D} = \mathrm{End}({}_\mathcal{A}\mathcal{M})$. Of course, $J_1(\mathcal{A}) = J(\mathcal{A})$ is the Jacobson radical of $\mathcal{A}$.

In our first theorem we consider derivations satisfying certain conditions which might appear somewhat special. However, it is considerably more general than various conditions considered by other authors. In [21] Herstein initiated the study of derivations d of prime rings such that $(a^d)^n = 0$ for all $a \in \mathcal{A}$, and such that $(a^d)^n$ is central for all $a \in \mathcal{A}$. Here, n is a fixed positive integer. Actually, [21] considers inner derivations only, but extensions to arbitrary derivations came quickly [18, 22, 41]. Further, Felzenszwalb and Lanski [16] investigated derivations d with $(a^d)^{n(a)} = 0$, $a \in \mathcal{A}$, where the positive integer $n(a)$ is not fixed but depends on a. The other, apparently unrelated project was started by Bergen, Herstein and Lanski [5] who treated derivations which have, besides 0, only invertible values. Several extensions of this result were proved; in particular, Mauceri and Misso [36] dealt with derivations whose values are either invertible or nilpotent. Clearly, the condition treated in the following theorem includes all the conditions mentioned. Of course, the conclusion is not that strong as in the particular cases just listed.

THEOREM 7.2: *Let $\mathcal{A}$ be any ring and d be a derivation of $\mathcal{A}$. Suppose that for every $a \in \mathcal{A}$ there exist positive integers $n = n(a), m = m(a)$ such that $(a^d)^n\mathcal{A} \subseteq \mathcal{A}(a^d)^m + J(\mathcal{A})$. Then $\mathcal{A}^d \subseteq J_3(\mathcal{A})$. Moreover, d is $\mathcal{M}$-inner for each simple left $\mathcal{A}$-module $\mathcal{M}$ such that $\dim(\mathcal{M}_\mathcal{D}) > 1$ where $\mathcal{D} = \mathrm{End}({}_\mathcal{A}\mathcal{M})$.*

Proof: Pick any simple left $\mathcal{A}$-module $\mathcal{M}$ and let $\mathcal{D} = \mathrm{End}({}_\mathcal{A}\mathcal{M})$.

Assume first d is $\mathcal{M}$-outer. If $\dim(\mathcal{M}_\mathcal{D}) > 1$, then picking any $x, y \in \mathcal{M}$ linearly independent over $\mathcal{D}$, there is $a \in \mathcal{A}$ such that $a^d x = 0$, $a^d y = y$ and $ax = y$ (Theorem 4.6). But then $(a^d)^n ax = y$ while $\mathcal{A}(a^d)^m x = 0$ for all positive integers n, m, contradicting our assumption.

It remains to show that $\mathcal{A}^d\mathcal{M} = 0$ whenever d is $\mathcal{M}$-inner and $\dim(\mathcal{M}_\mathcal{D}) \geq 3$. Thus, let T: $\mathcal{M} \to \mathcal{M}$ be a differential transformation such that $[T, L_a] = L_{a^d}$, $a \in \mathcal{A}$. Suppose first that Tx and x are $\mathcal{D}$-dependent for every $x \in \mathcal{M}$. By Lemma 7.1 we have that there is $\lambda \in \mathcal{D}$ such that $Tx = x\lambda$ for all $x \in \mathcal{M}$. But then $a^d x = T(ax) - a(Tx) = (ax)\lambda - a(x\lambda) = 0$ for any $a \in \mathcal{A}$, $x \in \mathcal{M}$. Therefore, we may assume that there is $x \in \mathcal{M}$ such that $y = Tx$ and x are $\mathcal{D}$-independent. We claim that there is a nonzero $z \in \mathcal{M}$ such that $y \notin (Tz)\mathcal{D} + z\mathcal{D} + x\mathcal{D}$. Indeed, pick any $z_0 \notin y\mathcal{D} + x\mathcal{D}$ (such z_0 exists for $\dim(\mathcal{M}_\mathcal{D}) \geq 3$ by assumption). With no loss of generality we may assume that $y \in (Tz_0)\mathcal{D} + z_0\mathcal{D} + x\mathcal{D}$, say, $y = (Tz_0)\lambda + z_0\mu + x\nu$ for some $\lambda, \mu, \nu \in \mathcal{D}$. Clearly, $\lambda \neq 0$. Now set $z = z_0 - x\lambda^{-1}$. Of course, $z \neq 0$. Recall that T is a differential transformation, so that $T(x\lambda^{-1}) = (Tx)\lambda^{-1} + x(\lambda^{-1})^\gamma$ where γ is a derivation of $\mathcal{D}$. Whence we have

$$\begin{aligned} Tz &= Tz_0 - T(x\lambda^{-1}) \\ &= y\lambda^{-1} - z_0\mu\lambda^{-1} - x\nu\lambda^{-1} - y\lambda^{-1} - x(\lambda^{-1})^\gamma \\ &\in z_0\mathcal{D} + x\mathcal{D} = z\mathcal{D} + x\mathcal{D}. \end{aligned}$$

But then $y \notin (Tz)\mathcal{D} + z\mathcal{D} + x\mathcal{D} = z_0\mathcal{D} + x\mathcal{D}$ and so our claim is proved. Now pick $a, b \in \mathcal{A}$ such that $aTz = az = ax = 0$, $ay = -x$ and $bz = x$. Then $(a^d)^n bz = x$ while $\mathcal{A}(a^d)^m z = 0$ for all positive integers n, m. This clearly contradicts our initial assumption and so the theorem is proved. ∎

Every derivation d of a division ring satisfies $a^d\mathcal{A} = \mathcal{A}a^d$ for every $a \in \mathcal{A}$, and every inner derivation d of $M_2(F)$, F a field, satisfies $(a^d)^2\mathcal{A} = \mathcal{A}(a^d)^2$ for every $a \in \mathcal{A}$. This clarifies why $J_3(\mathcal{A})$ appears in the conclusion.

The condition treated in the next result also unifies several conditions treated in the literature. A well-known theorem of Posner from 1957 [42] considers derivations d such that $[a^d, a]$ is central for every $a \in \mathcal{A}$. This theorem was generalized in various ways, in particular, derivations satisfying certain Engel type conditions were studied. The most general result in this direction was obtained recently by Lanski [31] examining the condition $[[\ldots[[(a^{n_0})^d, a^{n_1}], a^{n_2}], \ldots], a^{n_k}] = 0$ where the n_i's are fixed positive integers. Concerned with generalizations of Herstein's hypercenter theorem [20], some related conditions were investigated by Felzenszwalb [14], Felzenszwalb and Giambruno [15] and finally Chuang [12] who treated the most general condition $[(a^{n(a)})^d, a^{n(a)}]_k = 0$ (here, $[x, y]_k$ is defined as $[x, y]_0 = x$ and for $k \geq 1$, $[x, y]_k = [x, y]_{k-1}y - y[x, y]_{k-1}$). The results in the present paper enable a consideration of a condition fairly more general than those just mentioned (which, however, has to yield a weaker conclusion).

THEOREM 7.3: *Let $\mathcal{A}$ be any ring and d be a derivation of $\mathcal{A}$. Suppose that for each $a \in \mathcal{A}$ there is a nonnegative integer $n = n(a)$ such that $a^n a^d \in \mathcal{A}a + J(\mathcal{A})$. Then $\mathcal{A}^d \subseteq J_2(\mathcal{A})$.*

Proof: Pick any simple left $\mathcal{A}$-module $\mathcal{M}$ such that $\dim(\mathcal{M}_{\mathcal{D}}) \geq 2$ where $\mathcal{D} = \mathrm{End}({}_{\mathcal{A}}\mathcal{M})$, and let us show that $\mathcal{A}^d\mathcal{M} = 0$.

Suppose first that d is $\mathcal{M}$-outer. Pick any $\mathcal{D}$-independent elements $x, y \in \mathcal{M}$. By Theorem 4.6 there is $a \in \mathcal{A}$ such that $ax = 0$, $ay = y$ and $a^d x = y$. Then $a^n a^d x = y$ for every positive integer n, while $(\mathcal{A}a + J(\mathcal{A}))x = 0$. This contradiction implies that d must be $\mathcal{M}$-inner. Thus, there exists a differential transformation T such that $[T, L_a] = L_{a^d}$ for all $a \in \mathcal{A}$. Suppose that x and Tx are $\mathcal{D}$-independent for some $x \in \mathcal{M}$. Then, by the Jacobson density theorem, there is $a \in \mathcal{A}$ such that $ax = 0$ and $a(Tx) = Tx$. But then $a^n a^d x = -Tx$ for any positive integer n. However, since $(\mathcal{A}a + J(\mathcal{A}))x = 0$, this is impossible. Therefore, x and Tx are dependent for every $x \in \mathcal{M}$. Using Lemma 7.1 and arguing as in the proof of Theorem 7.2 we see that this yields the desired conclusion. ∎

In general we cannot claim that $\mathcal{A}^d \subseteq J(\mathcal{A})$. After all, $\mathcal{A}$ could be commutative and so the condition of Theorem 7.3 would be trivially satisfied; but it is not true that every derivation of a commutative ring has the range in the radical. However, in Banach algebras this is true indeed (by a Banach algebra we shall always mean a complex algebra). This was proved for continuous derivations in 1955 by Singer and Wermer [48], and generalized to arbitrary derivations more recently by Thomas [49]. Of course, this result does not hold in noncommutative settings, for we have inner derivations. Anyway, in the literature one can find a number of noncommutative extensions of the Singer–Wermer (and Thomas) theorem (see, e.g., [7, 9, 11, 25, 33, 34, 35, 38, 43, 45, 47, 50, 51, 55] and references given there). In particular, many conditions under which a derivation of a noncommutative Banach algebra has the range in the radical have been found. As a corollary to Theorem 7.3 we will now obtain a result of such type.

COROLLARY 7.4: *Let $\mathcal{A}$ be a Banach algebra and d be a continuous derivation of $\mathcal{A}$. Suppose that for each $a \in \mathcal{A}$ there is a nonnegative integer $n = n(a)$ such that $a^n a^d \in \mathcal{A}a + J(\mathcal{A})$. Then $\mathcal{A}^d \subseteq J(\mathcal{A})$.*

Proof: All we need to show is that $\mathcal{A}^d \mathcal{M} = 0$ for any simple left $\mathcal{A}$-module $\mathcal{M}$ which is one-dimensional over $\mathcal{D} = \mathrm{End}({}_{\mathcal{A}}\mathcal{M})$ (which is really a complex field [6, Corollary 5, p. 128]). But this is true for any continuous derivation of a Banach algebra. Namely, by Sinclair's theorem [47], $\mathrm{ann}(\mathcal{M})$ is invariant under d. Therefore, d induces a derivation on the algebra $\mathcal{A}/\mathrm{ann}(\mathcal{M})$ (defined by $a + \mathrm{ann}(\mathcal{M}) \mapsto a^d + \mathrm{ann}(\mathcal{M})$). However, $\mathcal{A}/\mathrm{ann}(\mathcal{M})$ is isomorphic to the complex field and so the induced derivation is trivial. That is, $\mathcal{A}^d \subseteq \mathrm{ann}(\mathcal{M})$ and the proof is complete. ∎

Let us point out a special case of Corollary 7.4 concerned with the Engel condition.

COROLLARY 7.5: *Let $\mathcal{A}$ be a Banach algebra and d be a continuous derivation of $\mathcal{A}$. Suppose that for each $a \in \mathcal{A}$ there is a nonnegative integer $n = n(a)$ such that $[a^d, a]_n \in J(\mathcal{A})$. Then $\mathcal{A}^d \subseteq J(\mathcal{A})$.*

Corollary 7.5 generalizes certain results in [11, 34, 55] and can be viewed as an analytic analogue of Lanski's result [30]; however, unlike in [30], n is not fixed, just as in the classical Engel theorem.

In the proof of Corollary 7.4, the continuity of d has been used at one place only, when applying Sinclair's theorem on the invariance of primitive ideals. The so-called noncommutative Singer–Wermer conjecture asks whether this theorem holds without assuming the continuity. This problem seems to be extremely difficult. Even in the classical commutative case this conjecture was open for more than three decades and was finally solved by Thomas [49]. Clearly, if the conjecture turns out to be true then the assumption of continuity can be omitted in Corollary 7.4. Moreover, the problem whether the continuity assumption in Corollary 7.4 (or in Corollary 7.5) is superflous or not is in fact equivalent to the noncommutative Singer–Wermer conjecture; cf. [33, 45]. It is known that in order to settle this conjecture it suffices to treat the unitization of a radical algebra [45], and hence one arrives at one-dimensional modules. It is interesting that a purely algebraic approach presented here has led to the same problem.

We conclude by mentioning that we have already obtained some further applications of the results in the present paper to both ring theory and the theory of Banach algebras, but we plan to include them in a subsequent, a somewhat more technical paper.

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